# LECTURE 2. SCHUR'S LEMMA. MASCHKE'S THEOREM.

In this course we consider only finite-dimensional vector spaces. Please treat each proposition in this text as an exercise.

## 1. Morphisms of representations and Schur's Lemma

**Definition 1.1.** Assume that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations. $A \in \mathrm{Hom}(V, U)$ is called a morphism from $T$ to $S$ if

$$AT(g) = S(g)A, \quad \forall g \in G. \tag{1.1}$$

We will denote the set of morphisms from the representation $T$ to the representation $S$ by $\mathrm{Mor}(T, S)$ (by $\mathrm{End}(T)$ if $S = T$).

*Question* 1.2. How can we reformulate the definition of a morphism of representations if we consider those representations as $G$-modules?

*Question* 1.3. Is $\mathrm{End}(T)$ a subspace of $\mathrm{End}(V)$?

**Exercise 1.4.** Describe all morphisms between two one-dimensional complex representations of the group $\mathbb{Z}_n$.

**Definition 1.5.** Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations. They are called isomorphic if $\exists A \in \mathrm{Iso}(V, U)$ such that $A \in \mathrm{Mor}(T, S)$. In this case, $A$ is called an isomorphism between representations $T$ and $S$. The set of isomorphisms of representations $T$ and $S$ is denoted by $\mathrm{Iso}(T, S)$. The set of automorphisms of the representation $T$ is denoted by $\mathrm{Aut}(T)$.

**Lemma 1.6.** *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations, $A \in \mathrm{Mor}(T, S)$. Then:*
*a)* $\ker A = \{v \in V \mid Av = 0\} \subset V$ *is an invariant subspace under $T$;*
*b)* $\mathrm{Im}A = \{Av \mid v \in V\} \subset U$ *is an invariant subspace under $S$.*

**Proposition 1.7.** *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are irreducible representations, $A \in \mathrm{Mor}(T, S)$. Then either $A = 0$ or $A \in \mathrm{Iso}(T, S)$.*

**Proposition 1.8.** *(Schur's Lemma) Suppose that $T : G \to \mathrm{GL}(V)$ is an irreducible complex representation, and $A \in \mathrm{End}(T)$. Then $\exists \lambda \in \mathbb{C}$ such that $A = \lambda \mathrm{Id}$.*

*Question* 1.9. Is the claim of Schur's Lemma still true over $\mathbb{R}$?

**Definition 1.10.** A vector space $V$ over a field $\mathbb{F}$ with an operation $\cdot : V \times V \to V$ is called an algebra if the following properties hold:

(1) $a(b + c) = ab + ac, \quad \forall a, b, c \in V$;

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(2) $(b+c)a = ba + bc, \quad \forall a, b, c \in V$;
(3) $\lambda(ab) = (\lambda a)b = a(\lambda b), \quad \forall a, b \in V, \forall \lambda \in \mathbb{F}$.

An algebra $V$ is called associative if $a(bc) = (ab)c$, $\forall a, b, c \in V$. An associative algebra $V$ is called a division algebra if $(V \setminus \{0\}, \cdot)$ is a group.

*Question* 1.11. Let $T : G \to \mathrm{GL}(V)$ be an irreducible representation over $\mathbb{F}$. Does $\mathrm{End}(T)$ have to be a division algebra over $\mathbb{F}$?

*Question* 1.12. What are the possible finite-dimensional division algebras over $\mathbb{C}$?

*Question* 1.13. What are the possible finite-dimensional division algebras over $\mathbb{R}$?

**Exercise 1.14.** Construct an irreducible real representation of a finite group such that its algebra of endomorphisms is isomorphic to the division algebra of quaternions. You may pick any finite group you prefer.

## 2. Complete reducibility of representations.

**Definition 2.1.** Let $\mathbb{F}$ be a field. Let us denote by $\mathrm{char}\mathbb{F}$ (the characteristic of the field $\mathbb{F}$) the minimal positive integer $n$ such that

$$\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0. \tag{2.1}$$

If there is no such natural number, we will write $\mathrm{char}\mathbb{F} := 0$.

**Proposition 2.2.** *(Maschke) Suppose that a group $G$ is finite, $\mathbb{F}$ is a field, $V$ is a vector space over the field $\mathbb{F}$, $T : G \to \mathrm{GL}(V)$ is a representation, and* $\mathrm{char}\mathbb{F}$ *does not divide $|G|$. Then the representation $T$ is completely reducible.*

*Proof.* Let $U \subset V$ be an invariant subspace. Let $W \subset V$ be an arbitrary subspace such that $U \oplus W = V$. Define $P \in \mathrm{End}(V)$ by

$$P(u) := u, \quad \forall u \in U, \quad P(w) := 0, \quad \forall w \in W. \tag{2.2}$$

Now we will average $P$ over the group $G$. Let

$$P_0 := \frac{1}{|G|} \sum_{g \in G} T(g) P T(g^{-1}). \tag{2.3}$$

**Exercise 2.3.** Show that $P_0 \in \mathrm{End}(T)$.

Thus, $\ker(P_0)$ is invariant with respect to $T$.

**Exercise 2.4.** Show that

$$U \oplus \ker P_0 = V. \tag{2.4}$$

□

*Question* 2.5. Suppose that $T : G \to \mathrm{GL}(V)$ is a completely reducible representation and $V$ is expressed as a sum of minimal invariant subspaces:

$$V = \bigoplus_{i=1}^{m} V_i \tag{2.5}$$

Is such an decomposition unique up to permutation of summands?

**Exercise 2.6.** Suppose that the group $G$ is finite and $T : G \to \mathrm{GL}(\mathbb{C}^2)$ is a representation. Suppose $\exists g, h \in G$ such that $T(g)T(h) \neq T(h)T(g)$. Can the representation $T$ be not irreducible?

## 3. One-dimensional representations

Consider some examples first.

**Exercise 3.1.** The quaternion group $Q_8$ is generated by the elements $i, j, \varepsilon$ such that $\varepsilon^2 = e$, $\varepsilon i = i\varepsilon,\ \varepsilon j = j\varepsilon,\ i^2 = j^2 = \varepsilon,\ ij = \varepsilon ji$. Find all its one-dimensional complex representations.

**Exercise 3.2.** The dihedral group $D_n$ is generated by a rotation $a$ and a reflection $s$ such that $a^n = s^2 = e$, $sas = a^{-1}$. Find all one-dimensional complex representations of the group $D_n$. Note that the answer depends on the parity of $n$.

For a more general approach, let us recall useful constructions from group theory.

**Definition 3.3.** Let $G$ be a group. For any $a, b \in G$ define the commutator $[a, b]$ by $[a, b] := aba^{-1}b^{-1}$. Define the derived subgroup $[G, G]$ as the subgroup generated by the set of elements $\{[a, b] \mid a, b \in G\}$.

**Proposition 3.4.** *Let $G$ be a group. Then:*
*a) $[G, G]$ is a normal subgroup of $G$;*
*b) the quotient group $G/[G, G]$ is abelian;*
*c) the subgroup $[G, G]$ is contained in every normal subgroups $H$ of $G$ with the property that the quotient group $G/H$ is abelian.*

**Proposition 3.5.** *Assume $G$ is a group, and let $p : G \to G/[G, G]$ be the canonical projection. For every one-dimensional representation $\alpha : G/[G, G] \to \mathbb{F}^*$ we define the one-dimensional representation $\bar{\alpha} : G \to \mathbb{F}^*$ by $\bar{\alpha} = \alpha \circ p$. The map $\alpha \mapsto \bar{\alpha}$ is a bijection between the set of one-dimensional representations of the group $G/[G, G]$ and the set of one-dimensional representations of the group $G$.*

**Exercise 3.6.** Show that $[S_n, S_n] = A_n$, where $A_n := sgn^{-1}(1)$.

**Exercise 3.7.** Find all one-dimensional complex representations of the group $S_n$.

**Exercise 3.8.** Find all one-dimensional complex representations of the group $A_4$.

Relevant chapters from the textbooks:

- Fulton, Harris, Section 1.1;
- James, Liebeck, Chapters 3 - 5, 7 - 9;
- Vinberg, Sections 1.1 – 1.4, 2.1 – 2.3, 4.1 – 4.3.