LECTURE 4. TENSOR PRODUCT OF REPRESENTATIONS

Please treat each unproven proposition in this text as an exercise.

1. Dual Representation

Definition 1.1. Let $T : G \to \mathrm{GL}(V)$ be a representation over a field F. Define its dual representation $T^* : G \to \mathrm{GL}(V^*)$ by

$$\boxed{T^*(g) = (T(g^{-1}))^*.} \tag{1.1}$$

Exercise 1.2. Show that T^* is indeed a representation of the group G.

Exercise 1.3. Define $T : \mathbb{Z} \to \mathrm{GL}(\mathbb{R}^3)$ by

$$T(n) = \begin{pmatrix} 1 & n & \frac{n^2+n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that T is a representation. Find the matrix of $T^*(n)$ in the dual basis.

Exercise 1.4. Let $T : G \to \mathrm{GL}(V)$ be a representation. Is it always true that the representations T and $(T^*)^*$ are isomorphic?

Exercise 1.5. Let T be an irreducible representation of a group G. Is it possible that the representation T^* is not irreducible?

2. Tensor Product of Representations

Definition 2.1. Let $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ be representations over a field $\mathbb{F}$. We define the tensor product of representations T and S $T \otimes S : G \to \mathrm{GL}(V \otimes U)$ by the following equality:

$$\boxed{(T \otimes S)(g) := (T(g)) \otimes (S(g)).} \tag{2.1}$$

Exercise 2.2. Let T be a representation of a finite group G. Is it true that the representation $T \otimes T^*$ of the group G always has a one-dimensional invariant subspace?

Definition 2.3. Let $T : G \to \mathrm{GL}(V)$ be a representation. Define the symmetric square $S^2T : G \to \mathrm{GL}(S^2(V))$ and the exterior square $\Lambda^2 T : G \to \mathrm{GL}(\Lambda^2(V))$ of the representation T by

$$\boxed{(S^2T)(g) := S^2(T(g)), \quad (\Lambda^2 T)(g) := \Lambda^2(T(g)).} \tag{2.2}$$

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3. Complexification

Definition 3.1. Let V be a real vector space. Let us consider the field $\mathbb{C}$ as a vector space over $\mathbb{R}$. The tensor product $\mathbb{C} \otimes V$ (taken over $\mathbb{R}$) is also a real vector space. We will introduce a complex vector space structure on $\mathbb{C} \otimes V$ by setting

$$\lambda \cdot (z \otimes v) := (\lambda z \otimes v), \quad \forall \lambda, z \in \mathbb{C}, \forall v \in V. \tag{3.1}$$

Here, we define multiplication by a fixed λ on pairs $\mathbb{C} \times V$. Note that this map is bilinear; we then extend this map to the tensor product $\mathbb{C} \otimes V$ using its universal property. We also define addition to be the same as in the tensor product $\mathbb{C} \otimes V$. The resulting complex vector space is called the complexification of the real vector space V and is denoted by $V^{\mathbb{C}}$. Also define

$$V_0^{\mathbb{C}} := \{1 \otimes v \mid v \in V\}. \tag{3.2}$$

For every subspace U of the vector space V, denote by $U^{\mathbb{C}}$ the following subspace of $V^{\mathbb{C}}$:

$$U^{\mathbb{C}} := \left\{ \sum_k z_k \otimes u_k \mid z_k \in \mathbb{C}, u_k \in U \right\} = \mathbb{C} \otimes U \tag{3.3}$$

Exercise 3.2. Let V be a real vector space. Show that $\forall v \in V^{\mathbb{C}}\ \exists u, w \in V$ such that

$$v = 1 \otimes u + i \otimes w. \tag{3.4}$$

Exercise 3.3. Let V be a real vector space. Show that $V^{\mathbb{C}}$ is indeed a complex vector space. Also, prove that if $\dim V < \infty$, then the dimension of $V^{\mathbb{C}}$ over $\mathbb{C}$ equals $\dim V$. What can be said about $V_0^{\mathbb{C}}$ and its dimension?

Definition 3.4. Suppose that V is a real vector space and $A \in \mathrm{End}(V)$. We define the complexification $A^{\mathbb{C}} \in \mathrm{End}(V^{\mathbb{C}})$ of the linear operator A by

$$A^{\mathbb{C}}(z \otimes v) := z \otimes (Av). \tag{3.5}$$

Note that

$$\forall v \in V_0^{\mathbb{C}} \quad A^{\mathbb{C}} v \in V_0^{\mathbb{C}}, \tag{3.6}$$

and the restriction of $A^{\mathbb{C}}$ to $V_0^{\mathbb{C}}$ coincides with A. Thus, we can view $A^{\mathbb{C}}$ as an extension of the operator A defined on $V_0^{\mathbb{C}}$.

Example 3.5. Suppose that V is a real vector space, $\alpha = \{v_1, \ldots, v_n\}$ is a basis of V, and $A \in \mathrm{End}(V)$. Then the set $\beta = \{1 \otimes v_1, \ldots, 1 \otimes v_n\}$ is a basis of $V^{\mathbb{C}}$, and the corresponding matrices coincide:

$$[A^{\mathbb{C}}]_\beta = [A]_\alpha. \tag{3.7}$$

Proposition 3.6. *Suppose that V is a real vector space and $A \in \mathrm{End}(V)$. Then the characteristic polynomial of the linear operator $A^{\mathbb{C}}$ has real coefficients and is equal to χ_A.*

Example 3.7. Suppose that V is a complex vector space, $\gamma = \{v_1, \ldots, v_n\}$ is a basis of V, $D \in \mathrm{End}(V)$, and

$$[D]_\gamma = (a_k^l + i b_k^l)_{1 \le k,l \le n} = A + iB, \quad A, B \in \mathrm{Mat}(n \times n, \mathbb{R}). \tag{3.8}$$

Now let us consider V as a vector space over $\mathbb{R}$, denoted by $V^{\mathbb{R}}$ to avoid confusion. Then the map D is a linear operator on $V^{\mathbb{R}}$; denote it by $D^{\mathbb{R}}$. Then $\delta = \{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$ is a basis of $V^{\mathbb{R}}$. We get that

$$[D^{\mathbb{R}}]_\delta = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{3.9}$$

Proposition 3.8. *Let V be a finite-dimensional real vector space. Define* $\mathrm{Conj} \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *by*

$$\mathrm{Conj}(z \otimes v) := \overline{z} \otimes v. \tag{3.10}$$

Then:

a) $\mathrm{Conj}^2 = \mathrm{Id}$*;*

b) $\forall A \in \mathrm{End}(V) \quad \mathrm{Conj} \circ A^{\mathbb{C}} = A^{\mathbb{C}} \circ \mathrm{Conj}$*;*
c) We get that $\frac{1}{2}(\mathrm{Id} + \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *and it is a projection operator onto* $V_0^{\mathbb{C}}$*;*
d) We get that $\frac{1}{2i}(\mathrm{Id} - \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *and*

$$\mathrm{Im}\left(\frac{1}{2i}(\mathrm{Id} - \mathrm{Conj})\right) = V_0^{\mathbb{C}}; \tag{3.11}$$

e)

$$\mathrm{Im}\left(\frac{1}{2}(\mathrm{Id} + \mathrm{Conj})\right) \oplus \mathrm{Im}\left(\frac{1}{2}(\mathrm{Id} - \mathrm{Conj})\right) = (V^{\mathbb{C}})^{\mathbb{R}}. \tag{3.12}$$

Proposition 3.9. *Suppose that* V *is a real finite-dimensional vector space and* W *is a complex subspace of* $V^{\mathbb{C}}$*. Then the following conditions are equivalent:*

$$(\mathrm{Conj}(W) = W) \quad \Longleftrightarrow \quad (\textit{There exists a subspace } U \subset V \textit{ such that } W = U^{\mathbb{C}}). \tag{3.13}$$

Definition 3.10. For a representation $T : G \to \mathrm{GL}(V)$, we define its complexification $T^{\mathbb{C}} : G \to \mathrm{GL}(V^{\mathbb{C}})$ by

$$T^{\mathbb{C}}(g) := (T(g))^{\mathbb{C}}. \tag{3.14}$$

Proposition 3.11. *Let* $T : G \to \mathrm{GL}(V)$ *be an irreducible real representation of odd dimension. Then its complexification* $T^{\mathbb{C}}$ *is also irreducible.*

Exercise 3.12. Find an example of an irreducible real representation T such that $\dim T$ is even and $T^{\mathbb{C}}$ is an irreducible complex representation.

Exercise 3.13. Find an example of an irreducible real representation T such that $\dim T$ is even and $T^{\mathbb{C}}$ is not irreducible.

Definition 3.14. Let V be a real inner product space. Define an inner product on $V^{\mathbb{C}}$ by

$$\langle 1 \otimes x_1 + i \otimes y_1, 1 \otimes x_2 + i \otimes y_2 \rangle_{\mathbb{C}} := \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + i(\langle y_1, x_2 \rangle - \langle x_1, y_2 \rangle). \tag{3.15}$$

Exercise 3.15. Let V be a real inner product space. Show that $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ defines an inner product on the complex vector space $V^{\mathbb{C}}$. We will take it as the default inner product on the complexification $V^{\mathbb{C}}$.

Relevant textbooks chapters:

- James, Liebeck, Chapter 19;
- Vinberg, Sections 1.5, 3.1-3.3, 3.6, 3.7.