LECTURE 5. CHARACTER OF A REPRESENTATION

Please treat each unproven proposition in this text as an exercise.

1. Complexification

Definition 1.1. Let V be a real vector space. Let us consider the field $\mathbb{C}$ as a vector space over $\mathbb{R}$. The tensor product $\mathbb{C} \otimes V$ (taken over $\mathbb{R}$) is also a real vector space. We will introduce a complex vector space structure on $\mathbb{C} \otimes V$ by setting

$$\lambda \cdot (z \otimes v) := (\lambda z \otimes v), \quad \forall \lambda, z \in \mathbb{C}, \forall v \in V. \tag{1.1}$$

Here, we define multiplication by a fixed λ on pairs $\mathbb{C} \times V$. Note that this map is bilinear; we then extend this map to the tensor product $\mathbb{C} \otimes V$ using its universal property. We also define addition to be the same as in the tensor product $\mathbb{C} \otimes V$. The resulting complex vector space is called the complexification of the real vector space V and is denoted by $V^{\mathbb{C}}$. Also define

$$V_0^{\mathbb{C}} := \{1 \otimes v \mid v \in V\}. \tag{1.2}$$

For every subspace U of the vector space V, denote by $U^{\mathbb{C}}$ the following subspace of $V^{\mathbb{C}}$:

$$U^{\mathbb{C}} := \left\{ \sum_k z_k \otimes u_k \mid z_k \in \mathbb{C}, u_k \in U \right\} = \mathbb{C} \otimes U \tag{1.3}$$

Exercise 1.2. Let V be a real vector space. Show that $\forall v \in V^{\mathbb{C}}\ \exists u, w \in V$ such that

$$v = 1 \otimes u + i \otimes w. \tag{1.4}$$

Exercise 1.3. Let V be a real vector space. Show that $V^{\mathbb{C}}$ is indeed a complex vector space. Also, prove that if $\dim V < \infty$, then the dimension of $V^{\mathbb{C}}$ over $\mathbb{C}$ equals $\dim V$. What can be said about $V_0^{\mathbb{C}}$ and its dimension?

Definition 1.4. Suppose that V is a real vector space and $A \in \mathrm{End}(V)$. We define the complexification $A^{\mathbb{C}} \in \mathrm{End}(V^{\mathbb{C}})$ of the linear operator A by

$$A^{\mathbb{C}}(z \otimes v) := z \otimes (Av). \tag{1.5}$$

Note that

$$\forall v \in V_0^{\mathbb{C}} \quad A^{\mathbb{C}} v \in V_0^{\mathbb{C}}, \tag{1.6}$$

and the restriction of $A^{\mathbb{C}}$ to $V_0^{\mathbb{C}}$ coincides with A after identifying V with $V_0^{\mathbb{C}}$. Thus, we can view $A^{\mathbb{C}}$ as an extension of the operator A defined on V.

Example 1.5. Suppose that V is a real vector space, $\alpha = \{v_1, \ldots, v_n\}$ is a basis of V, and $A \in \mathrm{End}(V)$. Then the set $\beta = \{1 \otimes v_1, \ldots, 1 \otimes v_n\}$ is a basis of $V^{\mathbb{C}}$, and the corresponding matrices coincide:

$$[A^{\mathbb{C}}]_\beta = [A]_\alpha. \tag{1.7}$$

Proposition 1.6. *Suppose that V is a real vector space and $A \in \mathrm{End}(V)$. Then the characteristic polynomial of the linear operator $A^{\mathbb{C}}$ has real coefficients and is equal to χ_A.*

Date: October 5, 2026.

Example 1.7. Suppose that V is a complex vector space, $\gamma = \{v_1, \dots, v_n\}$ is a basis of V, $D \in \mathrm{End}(V)$, and

$$[D]_\gamma = (a_k^l + ib_k^l)_{1\le k,l\le n} = A + iB, \quad A, B \in \mathrm{Mat}(n \times n, \mathbb{R}). \tag{1.8}$$

Now let us consider V as a vector space over $\mathbb{R}$, denoted by $V^{\mathbb{R}}$ to avoid confusion. Then the map D is a linear operator on $V^{\mathbb{R}}$; denote it by $D^{\mathbb{R}}$. Then $\delta = \{v_1, \dots, v_n, iv_1, \dots, iv_n\}$ is a basis of $V^{\mathbb{R}}$. We get that

$$[D^{\mathbb{R}}]_\delta = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{1.9}$$

Proposition 1.8. *Let V be a finite-dimensional real vector space. Define* Conj $\in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *by*

$$\mathrm{Conj}(z \otimes v) := \overline{z} \otimes v. \tag{1.10}$$

Then:

a) $\mathrm{Conj}^2 = \mathrm{Id}$*;*

b) $\forall A \in \mathrm{End}(V) \quad \mathrm{Conj} \circ A^{\mathbb{C}} = A^{\mathbb{C}} \circ \mathrm{Conj}$*;*

c) We get that $\frac{1}{2}(\mathrm{Id} + \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *and it is a projection operator onto* $V_0^{\mathbb{C}}$*;*

d) We get that $\frac{1}{2i}(\mathrm{Id} - \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *and*

$$\mathrm{Im}\left(\frac{1}{2i}(\mathrm{Id} - \mathrm{Conj})\right) = V_0^{\mathbb{C}}; \tag{1.11}$$

e)

$$\mathrm{Im}\left(\frac{1}{2}(\mathrm{Id} + \mathrm{Conj})\right) \oplus \mathrm{Im}\left(\frac{1}{2}(\mathrm{Id} - \mathrm{Conj})\right) = (V^{\mathbb{C}})^{\mathbb{R}}. \tag{1.12}$$

Proposition 1.9. *Suppose that V is a real finite-dimensional vector space and W is a complex subspace of $V^{\mathbb{C}}$. Then the following conditions are equivalent:*

$$(\mathrm{Conj}(W) = W) \quad \Longleftrightarrow \quad (\textit{There exists a subspace } U \subset V \textit{ such that } W = U^{\mathbb{C}}). \tag{1.13}$$

Definition 1.10. For a real representation $T : G \to \mathrm{GL}(V)$, we define its complexification $T^{\mathbb{C}} : G \to \mathrm{GL}(V^{\mathbb{C}})$ by

$$T^{\mathbb{C}}(g) := (T(g))^{\mathbb{C}}. \tag{1.14}$$

Proposition 1.11. *Let $T : G \to \mathrm{GL}(V)$ be an irreducible real representation of odd dimension. Then its complexification $T^{\mathbb{C}}$ is also irreducible.*

Exercise 1.12. Find an example of an irreducible real representation T such that $\dim T$ is even and $T^{\mathbb{C}}$ is an irreducible complex representation.

Exercise 1.13. Find an example of an irreducible real representation T such that $\dim T$ is even and $T^{\mathbb{C}}$ is not irreducible.

Definition 1.14. Let V be a real inner product space. Define an inner product on $V^{\mathbb{C}}$ by

$$\langle 1 \otimes x_1 + i \otimes y_1, 1 \otimes x_2 + i \otimes y_2 \rangle_{\mathbb{C}} := \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + i(\langle y_1, x_2 \rangle - \langle x_1, y_2 \rangle). \tag{1.15}$$

Exercise 1.15. Let V be a real inner product space. Show that $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ defines an inner product on the complex vector space $V^{\mathbb{C}}$. We will take it as the default inner product on the complexification $V^{\mathbb{C}}$.

2. Character of a representation

Definition 2.1. Let $T : G \to \mathrm{GL}(V)$ be a representation over a field $\mathbb{F}$. Define the character $\chi_T : G \to \mathbb{F}$ of the representation T by

$$\chi_T(g) := \mathrm{Tr}(T(g)), \quad g \in G. \tag{2.1}$$

Proposition 2.2. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations. Then*
a) $\chi_T(e) = \dim(T)$;
b) $\chi_T(hgh^{-1}) = \chi_T(g)$, $\forall g, h \in G$;
c) χ_T is constant on the conjugacy classes of the group G;
d) $\chi_{T\oplus S} = \chi_T + \chi_S$;
e) $\chi_{T\otimes S} = \chi_T \cdot \chi_S$.

Exercise 2.3. Let $T : G \to \mathrm{GL}(V)$ be a complex representation of a finite group G. Show that

$$|\chi_T(g)| \leq \dim T, \quad \forall g \in G. \tag{2.2}$$

Exercise 2.4. Assume that $T : G \to \mathrm{GL}(V)$ is a complex representation. Can we express the character $\chi_{\Lambda^2 T}$ using χ_T?

From Maschke's Theorem, we deduce that every real or complex representation T of a finite group G is isomorphic to the sum of irreducible representations with some multiplicities:

$$T \cong \bigoplus_{S\in\widehat{G}} m_S S. \tag{2.3}$$

These multiplicities are uniquely determined by T.

Taking the character of both sides, we get the equality

$$\chi_T = \sum_{S\in\widehat{G}} m_S \chi_S. \tag{2.4}$$

3. Orthogonality relations

Definition 3.1. Suppose that $\mathbb{F}$ is a field and G is a finite group. Denote by $\mathbb{F}[G]$ the set of functions from G to $\mathbb{F}$. The set $\mathbb{F}[G]$ has a natural structure of a vector space over $\mathbb{F}$. When $\mathbb{F} = \mathbb{C}$, we introduce an inner product on $\mathbb{C}[G]$ via the formula

$$\langle f_1, f_2\rangle := \frac{1}{|G|}\sum_{x\in G} f_1(x)\overline{f_2(x)}. \tag{3.1}$$

Also, define

$$Z(\mathbb{C}[G]) := \{f \in \mathbb{C}[G] \,|\, f(gxg^{-1}) = f(x), \forall x, g \in G\}. \tag{3.2}$$

$Z(\mathbb{C}[G])$ is called the subspace of class functions on G.

The next lectures will be devoted to the proof of the following remarkable result.

Proposition 3.2. *Let G be a finite group. The set $\{\chi_T \,|\, T \in \widehat{G}\}$ is an orthonormal basis of $Z(\mathbb{C}[G])$.*

There are several corollaries of that claim that we are going to use.

Proposition 3.3. *Suppose that T and S are irreducible complex representations of a finite group G. Then*

$$\langle \chi_T, \chi_S\rangle = \frac{1}{|G|}\sum_{g\in G} \chi_T(g)\overline{\chi_S(g)} = \begin{cases} 0, & \text{if } T \text{ is not isomorphic to } S, \\ 1, & \text{otherwise.} \end{cases} \tag{3.3}$$

Proposition 3.4. *Suppose that T and S are irreducible complex representations of a finite group G. If $\chi_T = \chi_S$, then the representations T and S are isomorphic.*

Proposition 3.5. *Let T be a complex representation of a finite group G. If $\langle \chi_T, \chi_T \rangle = 1$, then T is irreducible.*

Proposition 3.6. *Assume that G is a finite group, $x, y \in G$. Then*

$$\sum_{T \in \widehat{G}} \chi_T(x)\overline{\chi_T(y)} = \begin{cases} \frac{|G|}{|C|}, & \text{if } x \text{ and } y \text{ belong to the same conjugacy class } C, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Proposition 3.7. *Let G be a finite group. Then the following identity holds:*

$$\sum_{T \in \widehat{G}} (\dim T)^2 = |G|. \tag{3.5}$$

Relevant textbook chapters:

- Fulton, Harris, Sections 2.1 – 2.4;
- James, Liebeck, Chapters 10-16, 18, 19;
- Vinberg, Sections 3-6.