PROBLEM SET 3, DUE OCTOBER 3

3. Decomposition of a Representation; One-dimensional Representations

Problem 3.1. *Suppose that* $T : \mathbb{Z}_4 \to \mathrm{GL}(\mathbb{R}^4)$ *is defined by*

$$T(1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Describe $\mathrm{End}(T)$.

Problem 3.2. *Suppose that* T *is a complex representation of the group* G *and*

$$T \cong \bigoplus_{S \in \widehat{G}} m_S S. \tag{3.1}$$

Show that

$$m_S = \dim(\mathrm{Mor}(S, T)) = \dim(\mathrm{Mor}(T, S)), \quad \forall S \in \widehat{G}. \tag{3.2}$$

Problem 3.3. *The dihedral group* D_n *is generated by a rotation* a *and a reflection* s *satisfying* $a^n = s^2 = e$, *and* $sas = a^{-1}$. *Find all 1-dimensional complex representations of the group* D_n. *Note that the answer depends on the parity of* n.

Problem 3.4. *Prove that* $[S_n, S_n] = A_n$, *where* $A_n := sgn^{-1}(1)$.

Problem 3.5. *Find all 1-dimensional complex representations of the group* S_n.

Problem 3.6. *Find all 1-dimensional complex representations of the group* A_4.

Problem 3.7. *Let* $\{T_1, \ldots, T_n\}$ *be the set of all pairwise non-isomorphic one-dimensional complex representations of* G. *Prove that this set forms a group with respect to the pointwise multiplication. Describe this group for* $G = \mathbb{Z}_m$ *and* S_k.

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