PROBLEM SET 4, DUE OCTOBER 9

4. Tensor Product of Representations

Problem 4.1. *Suppose that* $A, B \in \mathrm{End}(V)$ *and* $C, D \in \mathrm{End}(U)$. *Show that*

$$(A \otimes C)(B \otimes D) = (AB) \otimes (CD).$$

Problem 4.2. *Suppose that* $A \in \mathrm{End}(V)$ *and* $B \in \mathrm{End}(U)$. *Show that*

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A) \cdot \mathrm{Tr}(B).$$

Is there an analogous formula for the determinant?

Problem 4.3. *Let* $\dim V = n$. *Find a formula for* $\dim S^k(V)$ *for a nonnegative integer* K.

Problem 4.4. *Define a representation* T *of the symmetric group* S_n *on* $\mathbb{C}^n$ *by*

$$T(\sigma)(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$$

for $\sigma \in S_n$. *Find all non-trivial invariant subspaces of the representation* T.

Problem 4.5. *Define* $T : \mathbb{Z} \to \mathrm{GL}(\mathbb{R}^3)$ *by*

$$T(n) = \begin{pmatrix} 1 & n & \frac{n^2+n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that T *is a representation. Find the matrix of* $T^*(n)$ *with respect to the dual basis.*

Problem 4.6. *Let* $T : G \to \mathrm{GL}(V)$ *be a representation. Is it always true that the representations* T *and* $(T^*)^*$ *are isomorphic?*

Problem 4.7. *Let* T *be an irreducible representation of a group* G. *Is it possible that the representation* T^* *is not irreducible?*

Problem 4.8. *Let* T *be a representation of a finite group* G. *Is it true that the representation* $T \otimes T^*$ *of the group* G *always has a one-dimensional invariant subspace?*

Problem 4.9. *Suppose that* T *is a representation of a group* G *over a field* $\mathbb{F}$ *and* $\mathrm{char}\mathbb{F} \neq 2$. *Show that*

$$T^{\otimes 2} = \Lambda^2 T \oplus S^2 T.$$

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