Commutative algebra. Problems. Set 1.

1. The element $x \in A$ is called idempotent iff $x^2 = x$. Prove that the only idempotents in the local ring are 0 and 1.

2. Suppose that all proper ideals in A are prime. Prove that A is a field.

3. Prove theorem 1.2 from lecture notes.

4. A multiplicative set S is called saturated iff the following implication holds: $z \in S, z = xy \Rightarrow x \in S$ and $y \in S$. Prove that S is saturated $\Leftrightarrow A \setminus S = \cup \mathfrak{p}_\alpha$ for some set of prime ideals $\{\mathfrak{p}_\alpha\}$. For arbitrary multiplicative set S let $\tilde{S}$ be the minimal saturated multiplicative set containing S, then $\tilde{S} = \{x$ such that x divides some $s \in S\}$. Prove that $A \setminus \tilde{S}$ is a union of all prime ideals not meeting S. Prove that the natural homomorphism $S^{-1}A \to \tilde{S}^{-1}A$ is an isomorphism.

5. Suppose I is maximal among the non-principal ideals of A (i.e. any $J \supset I$ is principal. Prove that I is prime. (Hint: suppose $x, y \notin I$ while $xy \in I$ and prove first that the ideal $I : c \stackrel{\text{def}}{=} \{a \in A | ac \in I\}$, where c is a generator of $I + (x)$, is principal).