Introduction to commutative and homological algebra.

Fall 2020.

Prerequisites: basic linear algebra including tensor products, some experience in geometry and topology is desirable but not essential.

Starting from ancient times mahemathics developed in interaction between algebra and geometry, partly complementary, partly competitive. Generally algebra solves equations while geometry describes properties of manifolds. Nowadays no much difference exists since one of the main tools of modern mathemathics is algebraic geometry.

The key points of its history were introduction of coordinates by René Descartes and Pierre de Fermat in the 17th century and definition of abstract algebraic manifolds (André Weil) and schemes (Alexander Grothendieck) in the 20th century. Weil extended the geometrc framework from classical **R** and **C** to arbitrary fields while Grothendieck attracted general commutative rings.

The two most impressive mathematical achievements of the 20th century, namely, the proof of Weil conjectures for algebraic varietes over the finite fields by Pierre Deligne (1973) which remains the closest approximation to the Riemann Hypothesis up to date, and the proof of the Fermat's Last Theorem by Andrew Wiles (1995) are both based on the methods of algebraic geometry.

The subject of the suggested introduction to commutative and homological algebra course is not the algebraic geometry itself but rather the language in which it speaks. This language is more universal and is widely used in other branches of mathematics including but not restricted to number theory (which stimulated the commutative algebra much) and topology (where the cohomology groups were first introduced).

The course will be divided in three parts as follows.

1. Commutative algebra
2. Category theory
3. Homological algebra

Part 1. Commutative algebra.

This part suggests general concepts concerning commutative rings and modules over them. We will keep our eyes on both the algebraic objects (rings, ideals, extensions) and respective geometric ones (schemes, subschemes, morphisms) paying special attention to finitely generated algebras over the field (resp. algebraic varietes).

Proposed syllabus:

1.1. Commutative rings, modules, ideals, quotient rings. Zero divisors and nilpotents. Prime and maximal ideals. Radicals. Jacobson rings. Localisation. Affine schemes. Zariski topology.

1.2. Noetherian rings and modules. Nakayama's lemma. Hilbert Basis Theorem. Krull dimension.

1.3. Integer ring extensions. Going-up and going-down theorems. Normal rings.

1.4. Discretely valuated rings and Dedekind rings.

1.5. Finitely generated algebras over a field. Noether's normalization theorem. Zariski's lemma. Hilbert's Nullstellensatz. Affine algebraic varietes. Dimension as transcendence degree.

1.6. Tensor product of modules. Flat modules and flat extensions.

Part 2. Category theory.

This part contains a compact dictionary of (now classical) language of categories which is necessary to discuss cohomology.

Proposed syllabus:

2.1. Categories and functors. Natural transformations. Adjoint functors. Representable functors. Yoneda lemma. Universal objects. Limits and colimits.

2.2. Additive categories. Abelian categories. Exact functors. Freyd-Mitchell embedding theorem.

Part 3. Homological algebra.

Homological algebra describes deviations of functors between abelian categories from being exact (i.e. from sending exact sequences to exact ones) in the way which could be considered in some sense analogous to measuring the non-linearity of polynomial by calculating its derivatives. This could be done by replacing the objects (or the functor itself) by their resolvents consisting of objects/functors of more simple nature. We are going to describe the constuction in details. The general methods of calculation and some particular examples will also be discussed.

Proposed syllabus:
3.1. Complexes and homology. Homotopy. Cone of the morphism. The snake lemma.
3.2. Projective and injective objects. Resolvents. δ - functors and construction of derived functors. Acyclic resolvents. Ext functor.
3.3. Tensor product of modules and Tor functor.
3.4. Composition of derived functors. Spectral sequences.
3.5. Group cohomology. Lyndon-Hochschild-Serre spectral sequence.
3.6. Category of sheaves. Cech cohomology. Leray spectral sequence.
3.7. Koszul complex. Hilbert's syzygy theorem.
3.8. Derived categories.

Textbooks.

M. F. Atiyah, I. G. McDonald. Introduction to commutative algebra. Addison-Wesley, 1969.
J.S.Milne. A Primer of Commutative Algebra. Lecture notes. March 2020.
C.A.Weibel. An introduction to homological algebra. Cambridge University Press, 1997.
D.Eisenbud. Commutative Algebra: With a View Toward Algebraic Geometry. Springer, 1995.
S. I. Gelfand, Yu. I. Manin. Methods of homological algebra. Springer, 2003.

Grading.

The students will get approximately 50 problems for solution during the semester. There also will be a midterm examination and the final examination (4 hours each). These provide three marks. The final mark will be close to $0.9\times$ (the average of the two best of the above) $+0.1\times$ (the worst one).