ADDENDUM TO LECTURE 10

1. Preliminary Algebraic Definitions

Definition 1.1. Consider a set G with a map $\text{mult} : G \times G \to G$. We refer to such maps as binary operations. In this context, the image $\text{mult}(a, b)$ is denoted by ab. The pair (G, mult) is called a monoid if the following conditions are satisfied:

(1) $a(bc) = (ab)c, \quad \forall a, b, c \in G$;
(2) There exists a unique element $e \in G$ such that $ae = ea = a, \quad \forall a \in G$ (e is called the identity element).

Definition 1.2. A monoid G is called a group if for every element a in G, there exists a unique element b in G such that

$$ab = ba = e. \tag{1.1}$$

This element b is called the inverse element of a and is denoted by a^{-1}.

Definition 1.3. A group G is called abelian if

$$ab = ba, \ \forall a, b \in G. \tag{1.2}$$

In an abelian group, the group operation is often denoted by $+$, and the inverse element of an element a is denoted by $-a$.

Definition 1.4. A set $\mathbb{F}$ with operations of multiplication and addition is called a field if the following conditions are met:

(1) $(\mathbb{F}, +)$ forms an abelian group. Its identity element is denoted by 0.
(2) $(\mathbb{F} \setminus \{0\}, \cdot)$ forms an abelian group. Its identity element is denoted by 1.
(3) The distributive property holds: $a(b + c) = ab + ac$, $(a + b)c = ac + bc, \quad \forall a, b, c \in \mathbb{F}$.

Definition 1.5. Two fields $\mathbb{F}_1$ and $\mathbb{F}_2$ are called isomorphic if there exists a bijective map $\varphi : \mathbb{F}_1 \to \mathbb{F}_2$ such that

$$\varphi(x + y) = \varphi(x) + \varphi(y), \tag{1.3}$$
$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y), \quad \forall x, y \in \mathbb{F}_1. \tag{1.4}$$

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2. Ordered fields

Definition 2.1. A field $\mathbb{F}$ is called ordered if there exists a subset S of $\mathbb{F} \times \mathbb{F}$ (for any pair $(a, b) \in S$, we will write $a < b$) such that the following properties must hold:

(1) For every $a, b \in \mathbb{F}$, exactly one of the following conditions holds: either $a = b$, or $a < b$, or $b < a$;

(2) For all $a, b, c \in \mathbb{F}$:
$$(((a < b) \wedge (b < c)) \Longrightarrow (a < c)); \tag{2.1}$$

(3) For all $a, b, c \in \mathbb{F}$:
$$((a < b) \Longrightarrow (a + c < b + c)); \tag{2.2}$$

(4) For all $a, b, c \in \mathbb{F}$:
$$(((a < b) \wedge (c > 0)) \Longrightarrow (ac < bc)). \tag{2.3}$$

Definition 2.2. Two ordered fields $\mathbb{F}_1$ and $\mathbb{F}_2$ are called isomorphic if there exists a field isomorphism $\varphi : \mathbb{F}_1 \to \mathbb{F}_2$ such that
$$\forall a, b \in \mathbb{F}_1 : ((a < b) \Longleftrightarrow (\varphi(a) < \varphi(b))). \tag{2.4}$$

3. Characterization of the set of real numbers

Definition 3.1. Let $\mathbb{F}$ be an ordered field. We say that it satisfies the least upper bound property if for every non-empty subset $E \subset \mathbb{F}$ that is bounded above, there exists $\sup(\mathrm{E}) \in \mathbb{F}$.

In our previous discussions we have established the following result.

Proposition 3.2. *There exists an ordered field that satisfies the least upper bound property.*

We refer to this field as the set of real numbers and denote it by $\mathbb{R}$.

One can also show the following statement.

Proposition 3.3. *Every ordered field that satisfies the least upper bound property is isomorphic to $\mathbb{R}$ as an ordered field.*