LECTURE 5. MAPS OF SETS

Please consider every proposition in this text to be an exercise.

1. Composition of maps.

Definition 1.1. Let X and Y be sets. Then a Cartesian product $X \times Y$ is a set that consists of all possible pairs (x, y), where $x \in X, y \in Y$.

Definition 1.2. Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an element $x \in X$ and an element $y \in Y$, such that for every $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true. Then we define the map $f : X \to Y$ defined by P on the domain X and the range Y to be the object which, given any input $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x, f(x))$ is true.

Exercise 1.3. Give a definition of the graph of a map $f : X \to Y$ as a subset of $X \times Y$.

Definition 1.4. Two maps $f : X \to Y$, $g : X \to Y$ with the same domain and range are said to be equal, $f = g$, if and only if $f(x) = g(x)$ for all $x \in X$.

Note that the domain may be an empty set.

Definition 1.5. Let $f : X \to Y$ and $g : Y \to Z$ be two maps, such that the range of f is the same set as the domain of g. We then define the composition $g \circ f : X \to Z$ of g and f to be the map defined explicitly by the formula $(g \circ f)(x) := g(f(x))$. If the range of f does not match the domain of g, we leave the composition $g \circ f$ undefined.

Proposition 1.6. *The composition is associative, namely, let $f : Z \to W$, $g : Y \to Z$, and $h : X \to Y$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.*

Exercise 1.7. Do there exist the maps $f, g : X \to X$ such that

$$f \circ g \neq g \circ f? \tag{1.1}$$

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2. Injectivity, Surjectivity.

Definition 2.1. A map $f : X \to Y$ is called injective if $\forall x_1, x_2 \in X$

$$(f(x_1) = f(x_2)) \Longrightarrow (x_1 = x_2). \tag{2.1}$$

Definition 2.2. A map $f : X \to Y$ is called surjective if $\forall y \in Y \, \exists x \in X$ such that $f(x) = y$.

Definition 2.3. A map $f : X \to Y$ is called bijective it is both injective and surjective.

Exercise 2.4. Give an example of a map that is
a) surjective but not injective;
b) injective but not surjective;
c) neither injective nor surjective.

Definition 2.5. Suppose $f : X \to Y$ is a map, $Z \subset X$, and $W \subset Y$. A set

$$f(Z) := \{f(z) \mid z \in Z\} \tag{2.2}$$

is called the image of the set Z. A set

$$f^{-1}(W) := \{x \in X \mid f(x) \in W\} \tag{2.3}$$

is called the preimage (the inverse image) of the set W.

Exercise 2.6. Reformulate Definitions 2.1, 2.2 and 2.3 in terms of sets images and preimages.

Exercise 2.7. a) Does the composition of injective maps have to be injective?
b) Does the composition of surjective maps have to be surjective?
c) Does the composition of bijective maps have to be bijective?

Definition 2.8. Let X and Y be sets. Denote by Y^X the set of all maps from X to Y. Denote by 2^X the set of all subsets of X.

Exercise 2.9. Let $|X| = n$, $|Y| = m$. Find $|Y^X|$ and $|2^X|$.

Definition 2.10. Let $f : X \to Y$ be a bijection. Define the inverse map $f^{-1} : Y \to X$ by the following rule: consider an arbitrary $y \in Y$, by our assumption $\exists! x \in X$ such that $f(x) = y$, then let

$$f^{-1}(y) := x. \tag{2.4}$$

Exercise 2.11. Suppose that $f : X \to Y$ and $g : Y \to Z$ are bijections. Is it always true that:
a)

$$\left(f^{-1}\right)^{-1} = f? \tag{2.5}$$

b)

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}? \tag{2.6}$$

Chapters from the textbooks relevant to the lecture:

- Tao, Sections 3.3-3.5;
- Zorich, Section 1.3.1-1.3.3.