LECTURE 6. INTEGERS.

Please consider every proposition in this text to be an exercise.

1. Informal definitions.

Definition 1.1. An integer is an expression of the form $m - n$, where m and n are natural numbers. Two integers are considered equal, $m - n = k - l$, if and only if $m + l = k + n$. We denote the set of all integers by $\mathbb{Z}$.

Definition 1.2. The sum of two integers, $(m - n) + (k - l)$, $m, n, k, l \in \mathbb{N}$, is defined by the formula

$$(m - n) + (k - l) := (m + k) - (n + l). \tag{1.1}$$

The product of two integers, $(m - n) \cdot (k - l)$, is defined by

$$(m - n) \cdot (k - l) := (mk + nl) - (ml + nk). \tag{1.2}$$

Definition 1.3. Define the negation of an integer $m - n$, $m, n \in \mathbb{N}$, to be an integer $n - m$:

$$-(m - n) := n - m. \tag{1.3}$$

We say that an integer $m - n$ is positive iff it can be expressed as $k - 0$, $k \in \mathbb{N}, k > 0$, and we say it is negative iff it can be expressed as $0 - k$, $k \in \mathbb{N}, k > 0$.

2. Formal definitions.

Definition 2.1. Let us introduce on the set $\mathbb{N} \times \mathbb{N}$ the following candidate to an equivalence relation:

$$((m, n) \sim (k, l)) \iff (m + l = n + k). \tag{2.1}$$

We will denote the set of corresponding equivalence classes by $\mathbb{Z}$ and we will call it the set of integers.

Exercise 2.2. Show that it is indeed an equivalence relation, i.e. it is reflexive, symmetric, and transitive.

Definition 2.3. Define the addition on the set $\mathbb{N} \times \mathbb{N}$ by the formula

$$(m, n) + (k, l) := (m + k, n + l). \tag{2.2}$$

The product of two pairs from $\mathbb{N} \times \mathbb{N}$, $(m, n) \cdot (k, l)$, is defined by

$$(m, n) \cdot (k, l) := (mk + nl, ml + nk). \tag{2.3}$$

Exercise 2.4. Show that the addition of integers in Definition 2.3 is well-defined on the set of integers $\mathbb{Z}$. Namely, we need to show that if we pick any two equivalence classes $a, b \in \mathbb{Z}$ and representatives of those classes $(m, n) \in a, (k, l) \in b$, then the equivalence class of the sum $(m, n) + (k, l)$ does not depend on the choice of (m, n) and (k, l) but only of a and b.

Exercise 2.5. Show that the multiplication of integers in Definition 2.3 is well-defined on the set of integers $\mathbb{Z}$.

We will often write ab instead of the notation $a \cdot b$ for the product of integers a and b.

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Definition 2.6. Define the negation of a pair $(m, n) \in \mathbb{N} \times \mathbb{N}$, to be a pair (n, m):

$$-(m, n) := (n, m). \tag{2.4}$$

Exercise 2.7. Show that the notion of the negation is well-defined on the set of integers.

Definition 2.8. We say that an integer a is positive iff $\exists m \in \mathbb{N}, m > 0$ such that $(m, 0) \in a$, and we say it is negative iff $\exists n \in \mathbb{N}, n > 0$ such that $(0, n) \in a$.

Proposition 2.9. *Let a be an integer. Then exactly one of the following three statements is true:*
(a) a can be expressed as $(0, 0)$;
(b) a is positive;
(c) a is negative.

Proposition 2.10. *Let a, b and c be integers. Then we have*

$$a + b = b + a; \tag{2.5}$$
$$(a + b) + c = a + (b + c); \tag{2.6}$$
$$a + 0 = 0 + a = a; \tag{2.7}$$
$$a + (-a) = (-a) + a = 0. \tag{2.8}$$

Proposition 2.11. *Let a, b and c be integers. Then we have*

$$ab = ba; \tag{2.9}$$
$$(ab)c = a(bc); \tag{2.10}$$
$$a \cdot 1 = 1 \cdot a = a; \tag{2.11}$$
$$(a + b)c = ac + bc; \tag{2.12}$$
$$a(b + c) = ab + ac. \tag{2.13}$$

Proposition 2.12. *Let a and b be integers such that $ab = 0$. Then $a = 0$ or $b = 0$.*

Proposition 2.13. *If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.*

3. Ordering of the integers.

It is natural to identify the subset of non-negative integers with the set of natural numbers.

Definition 3.1. Let a and b be integers. We say that a is greater than or equal to b, and write $a \geq b$ or $b \leq a$, iff we have $a = b + n$ for some non-negative integer n. We say that a is strictly greater than b, and write $a > b$ or $b < a$, iff $a \geq b$ and $a \neq b$.

Proposition 3.2. *Let a and b be integers. Then exactly one of the statements $a > b$, $a < b$, or $a = b$ is true.*

Proposition 3.3. *Let a, b, c be integers. Then we have:*
a) $a > b$ if and only if $a - b$ is a positive natural number.
b) If $a > b$, then $a + c > b + c$.
c) If $a > b$ and c is positive, then $ac > bc$.
d) If $a > b$, then $-a < -b$.
e) If $a > b$ and $b > c$, then $a > c$.

Chapters from the textbooks relevant to the lecture:

- Tao, Section 4.1;
- Zorich, Section 2.2.2.