LECTURE 7. RATIONAL NUMBERS.

Please consider every proposition in this text to be an exercise.

1. Addition and multiplication.

Definition 1.1. Let us introduce on the set $\mathbb{Z}\times(\mathbb{N}\setminus\{0\})$ the following candidate to an equivalence relation:

$$((m,n)\sim(k,l)) \quad\Longleftrightarrow\quad (ml=nk). \tag{1.1}$$

We will denote the set of corresponding equivalence classes by $\mathbb{Q}$ and we will call it the set of rational numbers.

Exercise 1.2. Show that it is indeed an equivalence relation, i.e. it is reflexive, symmetric, and transitive.

Definition 1.3. Define the addition on the set $\mathbb{Z}\times(\mathbb{N}\setminus\{0\})$ by the formula

$$(m,n)+(k,l) := (ml+nk, nl). \tag{1.2}$$

The product of two pairs from $\mathbb{Z}\times(\mathbb{N}\setminus\{0\})$, $(m,n)\cdot(k,l)$, is defined by

$$(m,n)\cdot(k,l) := (mk, nl). \tag{1.3}$$

Exercise 1.4. Show that the addition of rational numbers in Definition 1.3 is well-defined on the set $\mathbb{Q}$.

Exercise 1.5. Show that the multiplication of rational numbers in Definition 1.3 is well-defined on the set $\mathbb{Q}$.

We will often write ab instead of the notation $a\cdot b$ for the product of rationals a and b.

Definition 1.6. Define the negation of a pair $(m,n)\in\mathbb{Z}\times(\mathbb{N}\setminus\{0\})$, to be a pair $(-m,n)$:

$$-(m,n) := (-m,n). \tag{1.4}$$

Exercise 1.7. Show that the notion of the negation is well-defined on the set of rationals.

Definition 1.8. Denote by $0\in\mathbb{Q}$ the equivalence class of $(0,1)\in\mathbb{Z}\times(\mathbb{N}\setminus\{0\})$. Also denote by 1 the equivalence class of $(1,1)\in\mathbb{Z}\times(\mathbb{N}\setminus\{0\})$.

Proposition 1.9. *Let $a,b,c\in\mathbb{Q}$. Then we have*

$$a+b=b+a; \tag{1.5}$$
$$(a+b)+c=a+(b+c); \tag{1.6}$$
$$a+0=0+a=a; \tag{1.7}$$
$$a+(-a)=(-a)+a=0. \tag{1.8}$$

For two rational numbers a and b we will write $a-b$ instead of $a+(-b)$.

Date: October 16, 2024.

Proposition 1.10. *Suppose that a, b and c are rational numbers. Then we have*

$$ab = ba; \tag{1.9}$$
$$(ab)c = a(bc); \tag{1.10}$$
$$a \cdot 1 = 1 \cdot a = a; \tag{1.11}$$
$$(a+b)c = ac + bc; \tag{1.12}$$
$$a(b+c) = ab + ac. \tag{1.13}$$

Proposition 1.11. *For every rational number a such that $a \neq 0$ there exists a rational number b satisfying*

$$ab = ba = 1. \tag{1.14}$$

2. Ordering of the rational numbers.

Definition 2.1. We say that a is positive iff $\exists m, n \in \mathbb{N}, m > 0, n > 0$ such that $(m, n) \in a$, and we say it is negative iff $\exists m, n \in \mathbb{N}, m > 0, n > 0$ such that $(-m, n) \in a$.

Proposition 2.2. *Let a be a rational number. Then exactly one of the following three statements is true:*
(a) $a = 0$;
(b) a is positive;
(c) a is negative.

Definition 2.3. Let a and b be rational naumbers. We say that a is strictly greater than b, and write $a > b$ or $b < a$, iff $a - b$ is positive. We say that a is greater or equal than b, and write $a \geq b$ or $b \leq a$, iff $a > b$ or $a = b$.

Proposition 2.4. *Let a and b be rational numbers. Then exactly one of the statements $a > b$, $a < b$, or $a = b$ is true.*

Proposition 2.5. *Let a, b and c be rational numbers. Then we have:*
a) If $a > b$ and $b > c$, then $a > c$.
b) If $a > b$, then $a + c > b + c$.
c) If $a > b$ and c is positive, then $ac > bc$.

Chapters from the textbooks relevant to the lecture:

- Tao, Section 4.2;
- Zorich, Section 2.2.2.