LECTURE 8. REAL NUMBERS.

Please consider every proposition in this text to be an exercise.

1. Cauchy sequences.

Definition 1.1. Define the absolute value of a rational number x as

$$|x| := \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{otherwise.} \end{cases} \tag{1.1}$$

Definition 1.2. Let m be an integer. A sequence of rational numbers

$$(a_n)_{n \geq m} \tag{1.2}$$

is a map a from the set $\{k \in \mathbb{Z} \mid k \geq m\}$ to the set of rational numbers $\mathbb{Q}$. But, we will write a_n instead of denoting the image of an integer n by $a(n)$. Usually, we will consider a sequence with $m = 1$ and denote it by (a_n).

Definition 1.3. Denote by $\mathbb{Z}_+$ the set $\{k \in \mathbb{Z} \mid k \geq 1\}$.

Definition 1.4. A sequence (a_n) is said to be bounded iff $\exists M \in \mathbb{Q}$ such that $|a_n| \leq M$, $\forall n \in \mathbb{Z}_+$.

Definition 1.5. A sequence (a_n) is called a Cauchy sequence iff

$$\forall \varepsilon \in \mathbb{Q},\ \varepsilon > 0,\ \exists k \in \mathbb{Z}_+ \text{ such that } \forall p, q \in \mathbb{Z}_+, p, q > k \text{ we have that } |a_p - a_q| < \varepsilon. \tag{1.3}$$

Proposition 1.6. *Every Cauchy sequence is bounded.*

2. Definition of real numbers.

Definition 2.1. Suppose that $(a_n), (b_n)$ are sequences of rational numbers, $\varepsilon \in \mathbb{Q}$, and $\varepsilon > 0$. We say that the sequence (a_n) is eventually ε-close to (b_n) iff

$$\exists k \in \mathbb{Z}_+ \text{ such that } \forall n \in \mathbb{Z}_+, n > k \text{ we have that } |a_n - b_n| < \varepsilon. \tag{2.1}$$

Definition 2.2. Consider the set of all Cauchy sequences of rational numbers Ω and introduce the following candidate to an equivalence relation:

$$((a_n) \sim (b_n)) \iff (\forall \varepsilon > 0\ (a_n) \text{ is eventually } \varepsilon\text{-close to } (b_n)). \tag{2.2}$$

We will denote the set of corresponding equivalence classes by $\mathbb{R}$ and we will call it the set of real numbers.

Exercise 2.3. Show that it is indeed an equivalence relation, i.e. it is reflexive, symmetric, and transitive.

3. Addition and multiplication.

Definition 3.1. Define the addition and the multiplication on the set Ω termwise:

$$(a_n) + (b_n) := (a_n + b_n), \tag{3.1}$$

$$(a_n) \cdot (b_n) := (a_n \cdot b_n). \tag{3.2}$$

Exercise 3.2. Show that the sum of two Cauchy sequences is a Cauchy sequence.

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Exercise 3.3. Show that the addition of real numbers in Definition 3.1 is well-defined on the set $\mathbb{R}$.

Exercise 3.4. Show that the product of two Cauchy sequences is a Cauchy sequence.

Exercise 3.5. Show that the multiplication of real numbers in Definition 3.1 is well-defined on the set $\mathbb{R}$.

We often write ab instead of the notation $a \cdot b$ for the product of real numbers a and b.

Definition 3.6. Define the termwise negation of a sequence $(a_n) \in \Omega$:

$$-(a_n) := (-a_n). \tag{3.3}$$

Exercise 3.7. Show that the notion of the negation is well-defined on the set of real numbers.

Definition 3.8. Denote by $0 \in \mathbb{R}$ the equivalence class of $(0, 0, 0, \dots) \in \Omega$. Also denote by $1 \in \mathbb{R}$ the equivalence class of $(1, 1, 1, \dots) \in \Omega$.

Proposition 3.9. *Let $x, y, z \in \mathbb{R}$. Then we have*

$$x + y = y + x; \tag{3.4}$$
$$(x + y) + z = x + (y + z); \tag{3.5}$$
$$x + 0 = 0 + x = x; \tag{3.6}$$
$$x + (-x) = (-x) + x = 0. \tag{3.7}$$

For two real numbers x and y we will write $x - y$ instead of $x + (-y)$.

Proposition 3.10. *Suppose that x, y and z are real numbers. Then we have*

$$xy = yx; \tag{3.8}$$
$$(xy)z = x(yz); \tag{3.9}$$
$$x \cdot 1 = 1 \cdot x = x; \tag{3.10}$$
$$(x + y)z = xz + yz; \tag{3.11}$$
$$x(y + z) = xy + xz. \tag{3.12}$$

Exercise 3.11. Let $(a_n) \in x \in \mathbb{R}, x \neq 0$. Then $\exists k \in \mathbb{Z}_+$ such that $\forall n > k$ we have that $a_n \neq 0$.

Proposition 3.12. *For every real number x such that $x \neq 0$ there exists a real number y satisfying*

$$xy = yx = 1. \tag{3.13}$$

4. Examples of sequences.

Let us define the following sequences:

$$a_n := 0, \tag{4.1}$$

$$b_n := \frac{1}{n}, \tag{4.2}$$

$$c_n := (-1)^n \frac{1}{n}, \tag{4.3}$$

$$d_n := \frac{1}{n^2}, \tag{4.4}$$

$$e_n := 2, \tag{4.5}$$

$$f_n := 2 + (-1)^n \frac{1}{n}, \tag{4.6}$$

$$g_n := (-1)^n n, \tag{4.7}$$

$$h_n := (-1)^n n + \frac{1}{n}, \tag{4.8}$$

$$i_n := \frac{1}{n}, \text{ if } n \text{ is odd, } i_n := 0, \text{ if } n \text{ is even,} \tag{4.9}$$

$$j_n := \frac{1}{n}, \text{ if } n \text{ is odd, } j_n := 2, \text{ if } n \text{ is even,} \tag{4.10}$$

$$k_n := \frac{1}{n}, \text{ if } n \text{ is odd, } k_n := n, \text{ if } n \text{ is even.} \tag{4.11}$$

Exercise 4.1. Which of those sequences are Cauchy sequences?

Exercise 4.2. Which pairs of those sequences are equivalent?

Chapters from the textbooks relevant to the lecture:

- Tao, Sections 5.1 - 5.3.