LECTURE 9. ORDERING REAL NUMBERS.

Please consider every proposition in this text to be an exercise.

1. Properties of ordering.

Definition 1.1. An element $x \in \mathbb{R}$ is called positive iff there exists a sequence $(a_n) \in x$ and $c \in \mathbb{Q}, c > 0$ such that $\forall n \in \mathbb{Z}_+$ we have the inequality $a_n > c$. An element $x \in \mathbb{R}$ is called negative iff $-x$ is positive.

Proposition 1.2. *Let x be a real number. Then exactly one of the following three statements is true:*
(a) $x = 0$;
(b) x is positive;
(c) x is negative.

Proposition 1.3. *Suppose x and y are positive real numbers. Then $x + y$ is also positive.*

Proposition 1.4. *Suppose x and y are positive real numbers. Then xy is also positive.*

Definition 1.5. Suppose x and y are real numbers. We say that x is strictly greater than y, and write $x > y$ or $y < x$, iff $x - y$ is positive. We say that x is greater than or equal to y, and write $x \geq y$ or $y \leq x$, iff $x > y$ or $x = y$.

Proposition 1.6. *Let x and y be real numbers. Then exactly one of the statements $x > y$, $x < y$, or $x = y$ is true.*

Proposition 1.7. *Suppose that x, y and z are real numbers. Then we have:*
a) If $x > y$ and $y > z$, then $x > z$.
b) If $x > y$, then $x + z > y + z$.
c) If $x > y$ and z is positive, then $xz > yz$.

Definition 1.8. Define the absolute value of a real number x as

$$|x| := \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{otherwise.} \end{cases} \tag{1.1}$$

Proposition 1.9. *Suppose x and y are positive real numbers and $x > y$. Then x^{-1} and y^{-1} are also positive real numbers and $x^{-1} < y^{-1}$.*

Proposition 1.10. *Let (a_n) be a Cauchy sequence of non-negative rational numbers. Then it belongs to the equivalence class of a non-negative real number.*

Proposition 1.11. *Suppose (a_n) and (b_n) are Cauchy sequences of rational numbers such that*

$$a_n \geq b_n, \quad \forall n \in \mathbb{Z}_+, \tag{1.2}$$

and they belong to real numbers x and y, respectively. Then $x \geq y$.

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2. Archimedean property.

Proposition 2.1. *Let x be a positive real number. Then there exists $a \in \mathbb{Q}$ and $n \in \mathbb{N}$ such that*

$$a \leq x \leq n. \tag{2.1}$$

Proposition 2.2. *Let x and ε be positive real numbers. Then there exists a positive integer M such that*

$$M \cdot \varepsilon > x. \tag{2.2}$$

Proposition 2.3. *Suppose $x, y \in \mathbb{R}$ and $x < y$. Then there exists $q \in \mathbb{Q}$ such that*

$$x < q < y. \tag{2.3}$$

Chapters from the textbooks relevant to the lecture:

- Tao, Section 5.4.