LECTURE 10. THE LEAST UPPER BOUND.

Please consider every proposition in this text to be an exercise.

1. Definitions.

Definition 1.1. Let E be a subset of $\mathbb{R}$. A number $c \in \mathbb{R}$ is called an upper bound of the set E iff

$$x \leq c, \quad \forall x \in E. \tag{1.1}$$

Exercise 1.2. Does every set have at least one upper bound?

Exercise 1.3. Does there exist a set that has exactly one upper bound? Exactly two upper bounds?

Exercise 1.4. Does an empty set have an upper bound?

Definition 1.5. Suppose a subset E of the set $\mathbb{R}$ has an upper bound. A real number s is called the least upper bound of the set E (or, the supremum of E) iff it satisfies two conditions:
1) s is an upper bound of E,
2) If $c \in \mathbb{R}$ is an upper bound of E then $c \geq s$. In this case, we denote the number s by sup(E).

Proposition 1.6. *Every subset of the set $\mathbb{R}$ has at most one supremum.*

2. Examples of subsets of $\mathbb{R}$.

Let us consider the following subsets of the set of real numbers:

$$A := \{-n \mid n \in \mathbb{N}\}, \tag{2.1}$$

$$B := \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}, \tag{2.2}$$

$$C := \{-\frac{1}{n} \mid n \in \mathbb{Z}_+\}, \tag{2.3}$$

$$D := \{x \in \mathbb{R} \mid x < 1\}, \tag{2.4}$$

$$E := \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}, \tag{2.5}$$

$$F := \mathbb{N} \tag{2.6}$$

$$G := B \cup C. \tag{2.7}$$

Exercise 2.1. Which of those subsets are upper-bounded?

Exercise 2.2. For every upper-bounded subset from this list, find its supremum. Does it belong to the corresponding set?

3. Least Upper bound property.

Lemma 3.1. *Let E be a non-empty upper-bounded subset of $\mathbb{R}$. Then for every $n \in \mathbb{Z}_+$ there exists a unique $m_n \in \mathbb{Z}$ such that:*

$$\frac{m_n}{n} \text{ is an upper bound of } E; \tag{3.1}$$

$$\frac{m_n - 1}{n} \text{ is not an upper bound of } E. \tag{3.2}$$

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Proposition 3.2. *Let E be a non-empty upper-bounded subset of $\mathbb{R}$. Then there exists* $\sup(\mathrm{E}) \in \mathbb{R}$.

Proof. We will obtain the result from the following lemmas.

Lemma 3.3. *Show that the sequence*

$$\left(\frac{m_n}{n}\right) \tag{3.3}$$

defined in Lemma 3.1 is a Cauchy sequence.

Lemma 3.4. *Show that the sequence*

$$\left(\frac{m_n - 1}{n}\right) \tag{3.4}$$

defined in Lemma 3.1 is a Cauchy sequence.

Lemma 3.5. *Show that the sequences*

$$\left(\frac{m_n}{n}\right) \text{ and } \left(\frac{m_n - 1}{n}\right) \tag{3.5}$$

are equivalent.

Lemma 3.6. *Show that the real number defined by the sequence*

$$\left(\frac{m_n}{n}\right), \tag{3.6}$$

is an upper bound of E.

Lemma 3.7. *Show that the real number defined by the sequence*

$$\left(\frac{m_n - 1}{n}\right), \tag{3.7}$$

is not greater than any upper bound of E.

□

Exercise 3.8. Give definitions of a lower bound and the greatest lower bound (or, the infimum, denoted by inf).

Proposition 3.9. *Let E be a non-empty subset of $\mathbb{R}$. Define the subset*

$$-E := \{x \in \mathbb{R} \mid -x \in E\} \tag{3.8}$$

Then E is lower-bounded iff $-E$ is upper-bounded. In this case, $\inf(\mathrm{E})$ *exists and*

$$\inf(\mathrm{E}) = -\sup(-\mathrm{E}). \tag{3.9}$$

Definition 3.10. If E is not upper-bounded then define $\sup(\mathrm{E}) := +\infty$, $\inf(\mathrm{E}) := -\infty$. If E is an empty set, we state that $\sup(\mathrm{E}) := -\infty$, $\inf(\mathrm{E}) := +\infty$.

Chapters from the textbooks relevant to the lecture:

- Tao, Section 5.5.