LECTURE 11. REAL EXPONENTIATION, PART 1.

Please consider every proposition in this text to be an exercise.

1. Integer powers.

Definition 1.1. Let x be a real number. To raise x to the power 0, we define

$$x^0 := 1 \tag{1.1}$$

Now suppose recursively that x^n has been defined for some natural number n, then we define

$$x^{n+1} := x^n \cdot x. \tag{1.2}$$

If $m \in \mathbb{Z}$ and $m < 0$ we define

$$x^m := \frac{1}{x^{-m}}. \tag{1.3}$$

Proposition 1.2. *Suppose x, y are real numbers, and m, n are integers. Then the following identities hold:*

$$x^n \cdot x^m = x^{n+m}, \tag{1.4}$$

$$(x^n)^m = x^{nm}, \tag{1.5}$$

$$x^n \cdot y^n = (xy)^n. \tag{1.6}$$

Proposition 1.3. *Let $n \in \mathbb{Z}_+$. Then*

$$((x \in \mathbb{R}) \wedge (x^n = 0)) \quad \Longleftrightarrow \quad (x = 0). \tag{1.7}$$

Proposition 1.4. *Suppose $x > y \geq 0$ are real numbers. If n is a positive integer then $x^n > y^n$. If n is a negative integer then $x^n < y^n$.*

Proposition 1.5. *Let x be a real number. We have that*

$$|x^n| = |x|^n, \quad \forall n \in \mathbb{Z}. \tag{1.8}$$

2. Existence of n-th roots.

Proposition 2.1. *Suppose x is a positive real number and n is a positive integer. Then there exists a positive real number y such that $y^n = x$.*

Proof. We will deduce this result from the following lemmas.

Lemma 2.2. *Suppose x is a positive real number and n is a positive integer. Define*

$$E(x, n) := \{z \in \mathbb{R} \mid z^n \leq x\}. \tag{2.1}$$

Then the set $E(x, n)$ is upper-bounded.

Lemma 2.3. *Suppose that x, y are positive real numbers and n is a positive integer such that $y^n < x$. Then $\exists \varepsilon \in \mathbb{R}, \varepsilon > 0$ such that*

$$(y + \varepsilon)^n < x. \tag{2.2}$$

Lemma 2.4. *Suppose that x, y are positive real numbers and n is a positive integer such that $y^n > x$. Then $\exists \varepsilon \in \mathbb{R}, \varepsilon > 0$ such that*

$$(y - \varepsilon)^n > x. \tag{2.3}$$

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Chapters from the textbooks relevant to the lecture:

- Tao, Section 5.6.