LECTURE 12. REAL EXPONENTIATION, PART 2.

Please consider every proposition in this text to be an exercise.

1. Rational powers.

Proposition 1.1. *Suppose $x, y \in \mathbb{R}$, $x \geq 0$, $y \geq 0$, $n \in \mathbb{Z}_+$ and $x^n = y^n$. Then $x = y$.*

Definition 1.2. In the previous lecture, we have shown that for every non-negative real number x and for every positive integer n, there exists a non-negative real number y such that $y^n = x$. This number y is called the n-th root of x and is denoted by

$$x^{\frac{1}{n}} := y. \tag{1.1}$$

Definition 1.3. Suppose $x \in \mathbb{R}$, $x > 0$ and $m \in \mathbb{Z}$, $n \in \mathbb{Z}_+$. Define the rational power of x by the following equality:

$$x^{\frac{m}{n}} := \left(x^{\frac{1}{n}}\right)^m. \tag{1.2}$$

Proposition 1.4. *Suppose that $m, k \in \mathbb{Z}$, $n, l \in \mathbb{Z}_+$ satisfy the condition*

$$\frac{m}{n} = \frac{k}{l}. \tag{1.3}$$

Then for every positive real number x, we have that

$$x^{\frac{m}{n}} = x^{\frac{k}{l}}. \tag{1.4}$$

Proposition 1.5. *Suppose x, y are real positive numbers, and p, q are rational numbers. Then*

$$x^p \cdot x^q = x^{p+q}. \tag{1.5}$$

Proposition 1.6. *Suppose x, y are real positive numbers, and p is a rational number. Then*

$$x^p \cdot y^p = (xy)^p. \tag{1.6}$$

Proposition 1.7. *Suppose x, y are real positive numbers, and p, q are rational numbers. Then*

$$(x^p)^q = x^{pq}. \tag{1.7}$$

Proposition 1.8. *Suppose $x > y > 0$ are real numbers. If p is a positive rational number, then $x^n > y^n$. If n is a negative rational number, then $x^n < y^n$.*

Proposition 1.9. *Suppose $x \in \mathbb{R}, x > 1$ and $p, q \in \mathbb{Q}$. Then*

$$(x^p > x^q) \quad \Longleftrightarrow \quad (p > q). \tag{1.8}$$

Proposition 1.10. *Let x be a positive real number. We have that*

$$|x^p| = |x|^p, \quad \forall p \in \mathbb{Q}. \tag{1.9}$$

Chapters from the textbooks relevant to the lecture:

- Tao, Section 5.6.

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