LECTURE 13. LIMITS OF SEQUENCES.

Please consider every proposition in this text to be an exercise.

1. Definitions.

Definition 1.1. A sequence of real numbers (a_n) is by definition a map

$$a : \mathbb{Z}_+ \to \mathbb{R}, \tag{1.1}$$

where we denote $a(n)$ by a_n.

We say that a sequence (a_n) converges to a real number L if $\forall \varepsilon \in \mathbb{R}, \varepsilon > 0$, $\exists k \in \mathbb{Z}_+$ such that $\forall m > k$ we have that $|a_m - L| < \varepsilon$.

Proposition 1.2. *If a sequence of real numbers converges to real numbers L_1 and L_2 then $L_1 = L_2$.*

Definition 1.3. If a sequence (a_n) converges to L then the number L is called the limit of the sequence (a_n) and is denoted as

$$L = \lim_{n\to\infty} a_n. \tag{1.2}$$

The sequences are divided into two classes of convergent and divergent sequences.

Note that the index n in the equality (1.2) is irrelevant and can be replaced by another variable.

Exercise 1.4. Suggest an example of a divergent sequence.

Proposition 1.5. *Let (q_n) be a Cauchy sequence of rational numbers that defines a real number x. Then*

$$x = \lim_{n\to\infty} q_n. \tag{1.3}$$

Exercise 1.6. Give a definition of a Cauchy sequence of real numbers.

Exercise 1.7. Does a convergent sequence of real numbers have to be a Cauchy sequence?

Definition 1.8. A sequence of real numbers (a_n) is called bounded if $\exists M \in \mathbb{R}$ such that $|a_n| \leq M$, $\forall n \in \mathbb{Z}_+$.

Proposition 1.9. *Show that every convergent sequence is bounded.*

2. Algebraic properties of sequences limits.

Proposition 2.1. *Let (a_n) and (b_n) be convergent real sequences. Then the sequence $(a_n + b_n)$ is also convergent and*

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n. \tag{2.1}$$

Proposition 2.2. *Suppose (a_n) is a convergent real sequence and $c \in \mathbb{R}$. Then the sequence (ca_n) is also convergent and*

$$\lim_{n\to\infty} (ca_n) = c \cdot \lim_{n\to\infty} a_n. \tag{2.2}$$

Proposition 2.3. *Let (a_n) and (b_n) be convergent real sequences. Then the sequence $(a_n - b_n)$ is also convergent and*

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n. \tag{2.3}$$

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Proposition 2.4. *Let (a_n) and (b_n) be convergent real sequences. Then the sequence $(a_n \cdot b_n)$ is also convergent and*

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n. \tag{2.4}$$

Proposition 2.5. *Suppose (b_n) is a convergent real sequence such that $b_n \neq 0$, $\forall n \in \mathbb{Z}_+$ and $\lim_{n\to\infty} b_n \neq 0$. Then the sequence $1/b_n$ is also convergent and*

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\lim_{n\to\infty} b_n}. \tag{2.5}$$

Proposition 2.6. *Suppose that (a_n) and (b_n) are convergent real sequences such that $b_n \neq 0$, $\forall n \in \mathbb{Z}_+$ and $\lim_{n\to\infty} b_n \neq 0$. Then the sequence a_n/b_n is also convergent and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}. \tag{2.6}$$

Proposition 2.7. *Suppose (a_n) is a convergent real sequence such that $a_n \geq 0$, $\forall n \in \mathbb{Z}_+$. Then*

$$\lim_{n\to\infty} a_n \geq 0. \tag{2.7}$$

Proposition 2.8. *Suppose that (a_n) and (b_n) are convergent real sequences such that $a_n \geq b_n$, $\forall n \in \mathbb{Z}_+$. Then*

$$\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n. \tag{2.8}$$

Chapters from the textbooks relevant to the lecture:

- Tao, Section 6.1,
- Zorich, Sections 3.1.1, 3.1.2,
- Abbott, Sections 2.2, 2.3.