LECTURE 14. LIMIT POINTS OF SEQUENCES.

Please consider every proposition in this text to be an exercise.

1. Definitions.

Definition 1.1. A sequence (a_n) is called increasing (decreasing) iff $a_{n+1} \geq a_n$ ($a_{n+1} \leq a_n$), $\forall n \in \mathbb{Z}_+$. A sequence is called monotone iff it is increasing or decreasing.

A sequence (a_n) is called strictly increasing (strictly decreasing) iff $a_{n+1} > a_n$ ($a_{n+1} < a_n$), $\forall n \in \mathbb{Z}_+$.

Proposition 1.2. *Let (a_n) be a bounded increasing sequence. Then (a_n) is convergent and*

$$\lim_{n\to\infty} a_n = \sup\{\mathrm{a_n} \mid \mathrm{n} \in \mathbb{Z}_+\}. \tag{1.1}$$

Note that the sequence elements (a_n) do not have to be distinct, and by $\{a_n \mid n \in \mathbb{Z}_+\}$, we denote an ordinary set, not a multiset.

Proposition 1.3 (Nested intervals property)**.** *Consider a sequence of closed intervals $I_n = [a_n, b_n]$, $a_n, b_n \in \mathbb{R}$ such that $I_{n+1} \subset I_n$, $\forall n \in \mathbb{Z}_+$, i. e. (a_n) is an increasing sequence and (b_n) is a decreasing sequence. Then*

$$\bigcap_{n\in\mathbb{Z}_+} I_n \neq \emptyset. \tag{1.2}$$

If

$$\lim_{n\to\infty} (b_n - a_n) = 0, \tag{1.3}$$

then

$$\bigcap_{n\in\mathbb{Z}_+} I_n \tag{1.4}$$

is a singleton.

Definition 1.4. Let (a_n) be a sequence of real numbers. For an arbitrary strictly increasing sequence of positive integers (x_k) let us define a new sequence

$$b_k := a_{x_k}. \tag{1.5}$$

The sequence (b_k) is called a subsequence of the sequence (a_n)

Definition 1.5. Let (a_n) be a sequence of real numbers. A real number c is called a limit point of the sequence (a_n) iff there exists a subsequence (b_k) of (a_n) such that

$$\lim_{k\to\infty} b_k = c. \tag{1.6}$$

Definition 1.6. Let (a_n) be a sequence of real numbers. A real number c is called a limit point of the sequence (a_n) iff $\forall \varepsilon > 0\, \forall m \in \mathbb{Z}_+\, \exists k > m$ such that $|a_k - c| < \varepsilon$.

Exercise 1.7. Show that Definition 1.5 and Definition 1.6 are equivalent.

Proposition 1.8 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a limit point.*

Chapters from the textbooks relevant to the lecture:

- Tao, Section 6.2,
- Zorich, Sections 3.1.2.

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