LECTURES 1-2. NATURAL NUMBERS. THE PEANO AXIOMS

Please consider every proposition in this text to be an exercise.

1. The Peano axioms.

In this lecture, we are going to introduce a formal definition of the set of natural numbers. We will consider this definition a part of the axioms of set theory, which we will formulate in the next lecture. Of course, we need to check that our line of arguments is not circular.

Axiom 1. *There exists a set of natural numbers $\mathbb{N}$ satisfying the following properties.*

For every element n of the set $\mathbb{N}$ we have defined a successor $n{+}{+}$ of n such that the next five conditions hold:

(1) $0 \in \mathbb{N}$;
(2) *if* $n \in \mathbb{N}$ *then* $n{+}{+} \in \mathbb{N}$;
(3) $\forall n \in \mathbb{N}\ n{+}{+} \neq 0$;
(4) *if* $n \neq m$ *are two distinct natural numbers then* $n{+}{+} \neq m{+}{+}$;
(5) *(Principle of mathematical induction) Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n{+}{+})$ is also true. Then $P(n)$ is true for every natural number n.*

Instead of writing the natural numbers as $0, 0{+}{+}, (0{+}{+}){+}{+}, ((0{+}{+}){+}{+}){+}{+}, \dots$ we will employ our usual notation:

$$\begin{aligned} 1 &:= 0{+}{+}, \\ 2 &:= (0{+}{+}){+}{+}, \\ 3 &:= ((0{+}{+}){+}{+}){+}{+}, \\ &\vdots \qquad\quad \vdots \end{aligned}$$

Exercise 1.1. Show that $4 \neq 1$.

2. Addition of natural numbers.

We will give a definition of this operation using the principle of mathematical induction.

Definition 2.1. We put:

$$\forall m \in \mathbb{N} \quad 0 + m := m; \tag{2.1}$$

$$\forall m, n \in \mathbb{N} \quad (n{+}{+}) + m := (n + m){+}{+}. \tag{2.2}$$

Exercise 2.2. Show that $\forall m, n \in \mathbb{N}$ their sum $n + m$ is uniquely determined by Definition 2.1 and it is also a natural number.

Exercise 2.3. Show that

$$\forall n \in \mathbb{N}, \quad n + 0 = n. \tag{2.3}$$

Exercise 2.4. Show that

$$\forall m, n \in \mathbb{N}, \quad n + (m{+}{+}) = (n + m){+}{+}. \tag{2.4}$$

Those equalities allow us to deduce three important properties of addition.

Date: September 12, 2024.

Proposition 2.5 (Commutativity)**.** *For every pair of natural numbers m and n, we have that*

$$n + m = m + n. \tag{2.5}$$

Proposition 2.6 (Associativity)**.** *We get that*

$$\forall k, m, n \in \mathbb{N}, \quad (k + m) + n = k + (m + n). \tag{2.6}$$

Proposition 2.7 (Cancellation law)**.** *Suppose that $k, m, n \in \mathbb{N}$ such that*

$$k + m = k + n. \tag{2.7}$$

Then we have $m = n$.

3. Ordering of the natural numbers

Definition 3.1. Let n and m be natural numbers. We say that n is greater than or equal to m, and write $n \geq m$ or $m \leq n$, iff $\exists k \in \mathbb{N}$ such that $n = m + k$. We say that n is strictly greater than m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$. We say that n is positive, iff $n > 0$.

Exercise 3.2. Suppose that $m, n \in \mathbb{N}$ and $m > 0$. Show that $n + m > 0$.

Exercise 3.3. Suppose that $m, n \in \mathbb{N}$ and $m + n = 0$. Then either $m = 0$ or $n = 0$.

Exercise 3.4. Show that for an arbitrary positive natural n there exists exactly one natural number m such that $m{+}{+} = n$.

Those statements allow us to prove well-known properties of the order. Each property is a separate exercise.

Proposition 3.5. *Let k, m and n be natural numbers. Then*

(1) *$k \geq k$;*
(2) *if $k \geq m$ and $m \geq n$ then $k \geq n$;*
(3) *if $k \geq m$ and $m \geq k$ then $k = m$;*
(4) *$m \geq n$ if and only if $m + k \geq n + k$;*
(5) *$m < n$ if and only if $m{+}{+} \leq n$;*
(6) *$m < n$ if and only if $\exists k \in \mathbb{N}, k > 0$, such that $n = m + k$.*

Now we are ready to attack a more challenging claim about the order.

Proposition 3.6. *Let m and n be natural numbers. Then exactly one of the following statements is true:*

$$m < n, \; m = n, \; \text{or } m > n. \tag{3.1}$$

The properties of order allow us to obtain a very useful, stronger version of the principle of induction.

Proposition 3.7. *Let k be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq k$, we have the following implication: if $P(n)$ is true for all natural numbers $k \leq n < m$, then $P(m)$ is also true. (In particular, this means that $P(k)$ is true since in this case, the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq k$.*

4. Multiplication of natural numbers.

Again we will give a definition of this operation using the principle of mathematical induction.

Definition 4.1. We put:

$$\forall m \in \mathbb{N} \quad 0 \cdot m := 0; \tag{4.1}$$

$$\forall m, n \in \mathbb{N} \quad (n{+}{+}) \cdot m := (n \cdot m) + m. \tag{4.2}$$

Exercise 4.2. Show that $\forall m, n \in \mathbb{N}$ their product $n \cdot m$ is uniquely determined by Definition 4.1 and it is also a natural number.

Proposition 4.3 (Commutativity)**.** *For every pair of natural numbers m and n, we have that*

$$n \cdot m = m \cdot n. \tag{4.3}$$

Exercise 4.4. Let n, m be natural numbers. Then $n \cdot m = 0$ if and only if at least one of n, m is equal to zero. In particular, if n and m are both positive, then $n \cdot m$ is also positive.

Next, we will adopt the standard notation nm for the product $n \cdot m$.

Proposition 4.5 (Distributive law)**.** *For any natural numbers k, m, n we have equalities*

$$k(m + n) = km + kn \tag{4.4}$$

and

$$(k + m)n = kn + mn. \tag{4.5}$$

Proposition 4.6. *(Associativity) For any natural numbers k, m, n we have*

$$(km)n = k(mn). \tag{4.6}$$

Proposition 4.7. *If k and m are natural numbers such that $k < m$, and n is positive, then $kn < mn$.*

Exercise 4.8. Let k, m, n be natural numbers such that $kn = mn$ and n is non-zero. Show that $k = m$.

Proposition 4.9. *Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.*

Chapters from the textbooks relevant to the lecture:

- Tao, Sections 2.1, 2.2, 2.3;
- Zorich, Section 2.2.1.