PROBLEM SET 5, DUE OCTOBER 3

5. Maps of sets

Problem 5.1. *Suppose that A, B and C are sets. Show that*

$$A \times (B \cup C) = (A \times B) \cup (A \times C); \tag{5.1}$$
$$A \times (B \cap C) = (A \times B) \cap (A \times C); \tag{5.2}$$
$$A \times (B \setminus C) = (A \times B) \setminus (A \times C). \tag{5.3}$$

Problem 5.2. *Suppose that A, B, C and D are sets.*
a) What can we say about the relation between the sets

$$(A \times B) \cup (C \times D) \text{ and } (A \cup C) \times (B \cup D)? \tag{5.4}$$

Is one a subset of another? Are they equal?
b) What can we say about the relation between the sets

$$(A \times B) \cap (C \times D) \text{ and } (A \cap C) \times (B \cap D)? \tag{5.5}$$

c) What can we say about the relation between the sets

$$(A \times B) \setminus (C \times D) \text{ and } (A \setminus C) \times (B \setminus D)? \tag{5.6}$$

Problem 5.3. *Suppose that A, B, C and D are sets and*

$$A \times B \subset C \times D. \tag{5.7}$$

Can we always claim that $A \subset C$ and $B \subset D$?

Problem 5.4. *Suppose that $f, g : X \to Y, h : Y \to Z$ are maps and*

$$h \circ f = h \circ g. \tag{5.8}$$

Which condition on h (bijectivity, injectivity, surjectivity, no condition required) would allow us to claim $f = g$?

Problem 5.5. *Suppose that $f : X \to Y, g : Y \to Z$ are maps and the map $g \circ f$ is injective. Do both maps f and g have to be injective? What will change if we replace the condition of injectivity with surjectivity?*

Problem 5.6. *Suppose that $f : X \to Y$ is a map, $A \subset X$, and $B \subset Y$.*
a) What can we say about the relation between the sets

$$f(f^{-1}(B)) \text{ and } B? \tag{5.9}$$

Is one a subset of another? Are they equal?
b) What can we say about the relation between the sets

$$f^{-1}(f(A)) \text{ and } A? \tag{5.10}$$

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Problem 5.7. *Suppose that $f : X \to Y$ is a map, and $A, B \subset X$.*
a) What can we say about the relation between the sets

$$f(A \cup B) \text{ and } f(A) \cup f(B)? \tag{5.11}$$

b) What can we say about the relation between the sets

$$f(A \cap B) \text{ and } f(A) \cap f(B)? \tag{5.12}$$

c) What can we say about the relation between the sets

$$f(A \setminus B) \text{ and } f(A) \setminus f(B)? \tag{5.13}$$

Problem 5.8. *Suppose that $f : X \to Y$ is a map, and $A, B \subset Y$.*
a) What can we say about the relation between the sets

$$f^{-1}(A \cup B) \text{ and } f^{-1}(A) \cup f^{-1}(B)? \tag{5.14}$$

b) What can we say about the relation between the sets

$$f^{-1}(A \cap B) \text{ and } f^{-1}(A) \cap f^{-1}(B)? \tag{5.15}$$

c) What can we say about the relation between the sets

$$f^{-1}(A \setminus B) \text{ and } f^{-1}(A) \setminus f^{-1}(B)? \tag{5.16}$$

Problem 5.9. *Let X, Y and Z be sets and let $p_X : X \times Y \to X, p_Y : X \times Y \to Y$ be defined by*

$$p_X((x, y)) := x; \quad p_Y((x, y)) := y, \quad \forall x \in X, \forall y \in Y. \tag{5.17}$$

Is it true that for every pair of maps $f : Z \to X$ and $g : Z \to Y$ there exists a unique map $h : Z \to X \times Y$ such that

$$p_X \circ h = f \text{ and } p_Y \circ h = g? \tag{5.18}$$