PROBLEM SET 13, DUE DECEMBER 12

13. Limits of sequences.

Problem 13.1. *Suppose that* (a_n) *and* (b_n) *are real sequences such that*

$$0 \leq a_n \leq b_n, \forall n \in \mathbb{Z}_+, \tag{13.1}$$

and the sequence (b_n) *converges to* 0*. Show that* (a_n) *converges to* 0 *as well.*

Problem 13.2 (The squeeze (or sandwich) theorem)**.** *Suppose that* $(a_n), (b_n)$ *and* (c_n) *are real sequences such that*

$$a_n \leq b_n \leq c_n, \forall n \in \mathbb{Z}_+, \tag{13.2}$$

the sequences (a_n) *and* (c_n) *are convergent and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n. \tag{13.3}$$

Show that (b_n) *is also convergent and*

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n. \tag{13.4}$$

Problem 13.3. *Suppose that* (a_n) *is a real sequence such that*

$$\lim_{n\to\infty} |a_n| = 0. \tag{13.5}$$

Show that

$$\lim_{n\to\infty} a_n = 0. \tag{13.6}$$

Problem 13.4. *Show that for every real number* L *there exist convergent real sequences* (a_n) *and* (b_n) *such that* $b_n \neq 0$, $\forall n \in \mathbb{Z}_+$,

$$\lim_{n\to\infty} b_n = 0, \tag{13.7}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L. \tag{13.8}$$

Problem 13.5. *Suppose that* (a_n) *and* (b_n) *are convergent real sequences such that* $b_n \neq 0$, $\forall n \in \mathbb{Z}_+$ *and*

$$\lim_{n\to\infty} b_n = 0. \tag{13.9}$$

Can the sequence

$$\left(\frac{a_n}{b_n}\right) \tag{13.10}$$

be bounded and divergent?

Problem 13.6. *Give definitions when a sequence converges to* $+\infty$, $-\infty$ *and* ∞*. To avoid confusion, we will continue calling those sequences divergent.*

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Problem 13.7. *Find which of the following sequences are convergent. If a sequence is convergent, find its limit.*

$$a_n := n; \tag{13.11}$$
$$b_n := (-1)^n n; \tag{13.12}$$
$$c_n := (-1)^n; \tag{13.13}$$
$$d_n := \frac{1}{n}; \tag{13.14}$$
$$e_n := \frac{(-1)^n}{n}; \tag{13.15}$$
$$f_n := n^{(-1)^n}; \tag{13.16}$$
$$g_n := \frac{n^2+3}{n^3+4n+1}; \tag{13.17}$$
$$h_n := \sqrt{n}; \tag{13.18}$$
$$i_n := n^{-\frac{1}{100}}; \tag{13.19}$$
$$j_n := \frac{n}{3^n}; \tag{13.20}$$
$$k_n := \frac{2^n}{n^3+2n}. \tag{13.21}$$

Problem 13.8. *Suppose that (a_n) is a convergent real sequence and $m \in \mathbb{Z}_+$. Define the sequence*

$$b_n := a_{n+m}. \tag{13.22}$$

Show that (b_n) is also convergent and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n. \tag{13.23}$$

Problem 13.9. *Define the sequence (a_n) by the following relations:*

$$a_1 := 1, \quad a_{n+1} = a_n + \frac{1}{a_n}. \tag{13.24}$$

Is the sequence (a_n) convergent?