LECTURE 1. MAPS OF SETS

Please consider every proposition in this text as an exercise.

1. Composition of maps

Definition 1.1. Let X and Y be sets. The Cartesian product $X \times Y$ is the set of all pairs (x, y), where $x \in X, y \in Y$.

Exercise 1.2. Suppose that A, B, C and D are non-empty sets. Is it true that

$$(A \times B \subset C \times D) \iff ((A \subset C) \wedge (B \subset D))? \tag{1.1}$$

Definition 1.3. Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an element $x \in X$ and an element $y \in Y$, such that for every $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true. Then we define the map $f : X \to Y$ defined by P on the domain X and the codomain Y to be the object which, given any input $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x, f(x))$ is true.

Note that the domain may be the empty set.

Definition 1.4. Two maps $f : X \to Z$, $g : Y \to Z$ are equal, $f = g$, if and only if $X = Y$ and $f(x) = g(x)$ for all $x \in X$.

If two maps differ only by their codomains then, formally speaking, it would be natural to call them not equal. However, this situation doesn't occur in practice because usually we have some freedom in changing the codomain, so it is not discussed in the previous definition.

Definition 1.5. Let $f : X \to Y$ and $g : Z \to T$ be two maps such that $Y \subset Z$. We then define the composition $g \circ f : X \to T$ of g and f to be the map defined explicitly by the formula $(g \circ f)(x) := g(f(x))$.

Proposition 1.6. *Composition is associative, that is, let $f : Z \to W$, $g : Y \to Z$, and $h : X \to Y$ be maps. Then $f \circ (g \circ h) = (f \circ g) \circ h$.*

Exercise 1.7. Do there exist maps $f, g : X \to X$ such that

$$f \circ g \neq g \circ f? \tag{1.2}$$

Exercise 1.8. Let X, Y and Z be sets and let $p_X : X \times Y \to X, p_Y : X \times Y \to Y$ be defined by

$$p_X(x, y) := x; \quad p_Y(x, y) := y, \quad \forall x \in X, \forall y \in Y. \tag{1.3}$$

Is it true that for every pair of maps $f : Z \to X$ and $g : Z \to Y$ there exists a unique map $h : Z \to X \times Y$ such that

$$p_X \circ h = f \text{ and } p_Y \circ h = g? \tag{1.4}$$

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2. Injectivity, Surjectivity

Definition 2.1. A map $f : X \to Y$ is called injective if for all $x_1, x_2 \in X$

$$(f(x_1) = f(x_2)) \Longrightarrow (x_1 = x_2). \tag{2.1}$$

Definition 2.2. A map $f : X \to Y$ is called surjective if for every $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.

Definition 2.3. A map $f : X \to Y$ is called bijective if it is both injective and surjective.

Exercise 2.4. Give an example of a map that is
a) surjective but not injective;
b) injective but not surjective;
c) neither injective nor surjective.

Exercise 2.5. a) Is the composition of injective maps necessarily injective?
b) Is the composition of surjective maps necessarily surjective?
c) Is the composition of bijective maps necessarily bijective?

Definition 2.6. Suppose $f : X \to Y$ is a map and $Z \subset X$. The restriction of the map f to the set Z is, by definition, a map

$$f|_Z : Z \to Y \tag{2.2}$$

defined by

$$f|_Z(z) := f(z), \quad \forall z \in Z. \tag{2.3}$$

Exercise 2.7. Show that for every map $f : X \to Y$ there exists a subset Z of the domain X such that $f|_Z$ is injective and

$$f|_Z(Z) = f(X). \tag{2.4}$$

What is an analog of this claim if we replace injectivity by surjectivity?

Exercise 2.8. Suppose that $f, g : X \to Y$ and $h : Y \to Z$ are maps and

$$h \circ f = h \circ g. \tag{2.5}$$

Which condition on h (bijectivity, injectivity, surjectivity, no condition required) would allow us to conclude that $f = g$?

Exercise 2.9. Suppose that $f : X \to Y$ and $g : Y \to Z$ are maps and the map $g \circ f$ is injective. Do both maps f and g have to be injective? What changes if we replace the condition of injectivity with surjectivity?