LECTURE 2. INVERSE MAP

1. Preimage of a Subset

Definition 1.1. Suppose $f : X \to Y$ is a map, $Z \subset X$, and $W \subset Y$. The set

$$f(Z) := \{f(z) \mid z \in Z\} \tag{1.1}$$

is called the image of the set Z. The set

$$f^{-1}(W) := \{x \in X \mid f(x) \in W\} \tag{1.2}$$

is called the preimage (the inverse image) of the set W.

Exercise 1.2. Suppose that $f : X \to Y$ is a map, $A \subset X$, and $B \subset Y$.
a) What can we say about the relation between the sets

$$f(f^{-1}(B)) \text{ and } B? \tag{1.3}$$

Is one a subset of another? Are they equal?
b) What can we say about the relation between the sets

$$f^{-1}(f(A)) \text{ and } A? \tag{1.4}$$

Exercise 1.3. Suppose that $f : X \to Y$ is a map, and $A, B \subset X$.
a) What can we say about the relation between the sets

$$f(A \cup B) \text{ and } f(A) \cup f(B)? \tag{1.5}$$

b) What can we say about the relation between the sets

$$f(A \cap B) \text{ and } f(A) \cap f(B)? \tag{1.6}$$

c) What can we say about the relation between the sets

$$f(A \setminus B) \text{ and } f(A) \setminus f(B)? \tag{1.7}$$

Exercise 1.4. Suppose that $f : X \to Y$ is a map, and $A, B \subset Y$.
a) What can we say about the relation between the sets

$$f^{-1}(A \cup B) \text{ and } f^{-1}(A) \cup f^{-1}(B)? \tag{1.8}$$

b) What can we say about the relation between the sets

$$f^{-1}(A \cap B) \text{ and } f^{-1}(A) \cap f^{-1}(B)? \tag{1.9}$$

c) What can we say about the relation between the sets

$$f^{-1}(A \setminus B) \text{ and } f^{-1}(A) \setminus f^{-1}(B)? \tag{1.10}$$

Exercise 1.5. Reformulate the definitions of injectivity, surjectivity and bijectivity in terms of set images and preimages.

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Definition 1.6. Let X and Y be sets. Denote by Y^X the set of all maps from X to Y. Denote by 2^X the set of all subsets of X.

2. Inverse Map

Exercise 2.1. Let $|X| = n$, $|Y| = m$. Find $|Y^X|$ and $|2^X|$. How many injective maps from X to Y are there?

Definition 2.2. For every set X define the identity map $\mathrm{Id}_X : X \to X$ by

$$\mathrm{Id}_X(x) := x, \quad \forall x \in X. \tag{2.1}$$

Exercise 2.3. Suppose that the maps $f, g : X \to X$ satisfy the equality

$$g \circ f = \mathrm{Id}_X. \tag{2.2}$$

Must the map f be injective? Must it be surjective? Must it be bijective? Consider both the cases when X is finite and when X is infinite.

Definition 2.4. Let $f : X \to Y$ be a bijection. Define the inverse map $f^{-1} : Y \to X$ by the following rule: consider an arbitrary $y \in Y$. By our assumption, there exists a unique $x \in X$ such that $f(x) = y$. We then let

$$f^{-1}(y) := x. \tag{2.3}$$

Note the confusing coincidence of the notation for the inverse map and the preimage of a set.

Exercise 2.5. Suppose that $f : X \to Y$ and $g : Y \to Z$ are bijections. Is it always true that:
a)

$$\left(f^{-1}\right)^{-1} = f? \tag{2.4}$$

b)

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}? \tag{2.5}$$

Relevant textbook chapters:

- Tao, Section 3.4;
- Zorich, Section 1.3.3.