MATH IN MOSCOW. COMMUTATIVE ALGEBRA. HOMEWORK 1

(1) Let A be a ring, and let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of A-modules. Prove that M_2 is Noetherean (resp., Artinian) if and only if M_1 and M_3 are Noetherean (resp., Artinian).

(2) Let A be a Noetherean ring. Prove that the ring of power series in one variable $A[[x]]$ is Noetherean.

(3) Prove that a module M is Noetherean if and only if any submodule of M is finitely generated.

(4) Construct an example of an Artinian ring that has infinitely many ideals.

(5) Let A be an Artinian ring. Assume that M is a finitely generated A-module. Then M is Artinian.

(6) Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals and let I be an ideal in a ring A. Assume that $I \subset \bigcup_{i=1}^{n} \mathfrak{p}_i$. Then $I \subset \mathfrak{p}_i$ for some i.