MATH IN MOSCOW. COMMUTATIVE ALGEBRA.
HOMEWORK 2

(1) Prove that the polynomial ring $\mathbb{K}[x]$ is integrally closed in the field $\mathbb{K}(x)$.

(2) Let A be a ring, and let

$$M_1 \to M_2 \to M_3 \to 0 \tag{1}$$

be a sequence of A-modules, and let N be an A-module. Prove that the induced sequence

$$0 \to \mathrm{Hom}(M_3, N) \to \mathrm{Hom}(M_2, N) \to \mathrm{Hom}(M_1, N)$$

is exact if and only the sequence (1) is exact.

(3) Prove a statement analogous to the previous one, but for $\mathrm{Hom}(N, -)$ instead of $\mathrm{Hom}(-, N)$.

(4) Show that an exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0 \tag{2}$$

does not, in general, induce an exact sequence

$$0 \to \mathrm{Hom}(M_3, N) \to \mathrm{Hom}(M_2, N) \to \mathrm{Hom}(M_1, N) \to 0.$$

Prove an analogous statement for $\mathrm{Hom}(N, -)$ instead of $\mathrm{Hom}(-, N)$.

(5) Consider a commutative diagram of A-modules with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{\phi_1} & M_2 & \xrightarrow{\phi_2} & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{\psi_1} & N_2 & \xrightarrow{\psi_2} & N_3 & \longrightarrow & 0 \end{array} \tag{3}$$

Prove that it induces an exact sequence of modules

$$0 \to \mathrm{Ker}(f_1) \to \mathrm{Ker}(f_2) \to \mathrm{Ker}(f_3) \to \mathrm{Coker}(f_1) \to \mathrm{Coker}(f_2) \to \mathrm{Coker}(f_3) \to 0.$$

(6) Construct an example of an algebra over a field which is Noetherean but not finitely generated.