MATH IN MOSCOW. COMMUTATIVE ALGEBRA.
HOMEWORK 5

(1) Let $f\colon A \to B$ be a ring homomorphism. Let $I \subset A$ and $J \subset B$ be ideals. Define the *extension* and the *restriction* (sometimes also called *contraction*) of ideals I and J as follows: $I^e = f(I)B$ and $J^r = f^{-1}(J)$. Prove that
   (a) $I \subset I^{er}$, $J \supset J^{re}$,
   (b) $I^r \subset I^{rer}$, $J^e \supset J^{ere}$.
   (c) extension commutes with taking sums,
   (d) restriction does not commute with taking sums.

(2) Let $S \subset A$ be a multiplicative subset, and let $f\colon A \to S^{-1}A$ be a homomorphism to the localization. Consider an ideal $I \subset A$. Prove that $I^e = (1)$ if and only if I intersects S.

(3) Let $A \subset B$ be a ring extension. Assume that $B \setminus A$ is a multiplicative subset. Prove that A is integrally closed in B.

(4) Let A be a ring, and G be a finite group that acts on A by automorphisms. Prove that A is integral over A^G, where $A^G \subset A$ is the ring of G-invariants, that is, elements $a \in A$ such that $g(a) = a$ for any $g \in G$.

(5) Let I, J, K be ideals in a ring A. Define the *ideal quotient* $(I : J)$ as the set
$$\{a \in A \mid aJ \subset I\}.$$
   Prove that
   (a) $I \subset (I : J)$,
   (b) $(I : J)J \subset I$,
   (c) $((I : J) : K) = (I : JK) = ((I : K) : J)$.