MATH IN MOSCOW. ALGEBRAIC GEOMETRY. HOMEWORK 1

(1) Give an example of a non-principal ideal in $\mathbb{C}[x, y]$ and in $\mathbb{Z}[x]$.

(2) Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be algebraic subsets. Prove that $V \times W \subset \mathbb{A}^{n+m}$ is an algebraic subset.

(3) In the assumptions of the previous problem, prove that if V and W are irreducible, then $V \times W$ is irreducible as well.

(4) Let I be an ideal in $\mathbb{K}[x_1, \dots, x_n]$. Define $V(I) = \{a \in \mathbb{A}^n \mid f(a) = 0, \forall f \in I\}$. Prove that for algebraic subsets W, W' and for ideals J, J' one has
- $W \subset V(I(W))$,
- $I \subset I(V(I))$,
- $W \subset W' \implies I(W) \supset I(W')$,
- $J \subset J' \implies V(J) \supset V(J')$.

(5) In the assumptions of the previous problem, give an example when $I \neq I(V(I))$.

(6) Prove that the subset $\{x_1 - e^{x_2} = 0\} \subset \mathbb{A}^2_{\mathbb{C}}$ is not algebraic.