MATH IN MOSCOW. ALGEBRAIC GEOMETRY. HOMEWORK 5

(1) Consider the projection π of a rank 3 conic $C \subset \mathbb{P}^2$ from a point $p \in C$ to a line $L \simeq \mathbb{P}^1 \subset \mathbb{P}^2$ such that $p \notin L$. Prove that π is an isomorphism.

(2) Prove that, over the field of rational numbers $\mathbb{Q}$, a rank 3 conic $C \subset \mathbb{P}^2_{\mathbb{Q}}$ is either empty, or isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$.

(3) Using problems 1 and 2, find a parametrisation for the set of Pythagorean triples, that is, integers (x, y, z) such that $x^2 + y^2 = z^2$.

(4) Consider the projection π of a smooth conic $C \subset \mathbb{P}^2$ from a point $p \notin C$ to a line $L \simeq \mathbb{P}^1 \subset \mathbb{P}^2$ such that $p \notin L$.

    How many preimages the general point on L has? How many *ramification points* (that is, points on L that have less preimages than a general point) the morphism π has?

(5) The same question as in Problem 4 for the projection from a point p on an *elliptic curve* (that is, smooth cubic curve in $\mathbb{P}^2$).

(6) Prove that over $\mathbb{C}$, a smooth curve $C \subset \mathbb{P}^2$ is a compact oriented topological surface. Hence, by the classification theorem in topology, it is a sphere with g handles. Find g for a smooth cubic curve. What about a smooth curve of arbitrary degree $d \geq 1$?