MATH IN MOSCOW. ALGEBRAIC GEOMETRY. HOMEWORK 6

(1) Prove that if $X = X_1 \cup X_2$, and $p \in X_1 \cap X_2$, then $T_pX_1 \cup T_pX_2 \subset T_pX$. Is it true that $T_pX_1 \cup T_pX_2 = T_pX$?

(2) Prove that for a quasi-projective variety X and a point $p \in X$, for the local ring $\mathcal{O}_{p,X}$ we have $\mathcal{O}_{p,X} = \bigcup_{p \in U} \mathbb{K}[U]$ where U are affine neighbourhoods of p.

(3) Prove that if X is smooth at the point $x \in X$ and Y is smooth at the point $y \in Y$, then $X \times Y$ is smooth at $(x, y) \in X \times Y$.

(4) * Prove that any rational map from a smooth curve to a projective variety can be extended to a regular map. Is it true for non-projective varieites?

(5) Give an example of a morphism with finite fibers which is not finite.

(6) Let X be a quadric cone, that is

$$X = \{x^2 + y^2 = z^2\} \subset \mathbb{A}^3.$$

Prove that X is normal.

(7) Let A be an integral domain. An element $a \in A$ is called *irreducible*, if it cannot be represented as a product of two non-invertible elements. An integral domain A is called *factorial*, if any element x can be represented in the form

$$x = up_1 \dots p_n, \qquad n \geq 0,$$

where u is an invertible element, elements p_i are irreducible, and such representation is unique up to permutation of p_i and multiplication by invertible elements. Prove that a factorial ring is integrally closed.