MATH IN MOSCOW. ALGEBRAIC GEOMETRY.
HOMEWORK 4

(1) Find all regular functions $\mathbb{P}^1 \to \mathbb{K}$.

(2) Let X be an affine algebraic set. Prove that the algebra of regular functions on X considered as a quasi-affine algebraic set is the same as the algebra of regular functions on X considered as an affine algebraic set.

(3) Prove that Zariski closed subsets of $\mathbb{P}^n$ are precisely the zero sets of some families of homogeneous polynomials in $n+1$ variables.

(4) By a *projective closure* of a subset X in $\mathbb{P}^n$ we mean the closure of X in the Zariski topology on $\mathbb{P}^n$. Write homogeneous equations for projective closures of the following subsets of $\mathbb{A}^2 \subset \mathbb{P}^2$. Write their equations in the other two affine charts. Draw pictures over $\mathbb{R}$.
  (a) $y = x^2$,
  (b) $y = x^3$,
  (c) $y^2 = x^3 + x^2$,
  (d) $x^2 + (y-1)^2 = 1$.

(5) Consider $\mathbb{R}^2$ as a real part of the standard affine chart $U_0 \subset \mathbb{CP}^2$. Find two points in $\mathbb{CP}^2$ such that any Euclidean circle in $\mathbb{R}^2$ contains them, if we consider it as a projective subset over $\mathbb{C}$.

(6) Let V be a $(n+1)$-dimensional vector space, and let $f\colon \mathbb{P}(V) \to \mathbb{P}(V)$ be a projective linear transformation that corresponds to an element $A \in \mathrm{GL(n+1, \mathbb{K})}$. Assume that all fixed points of f are isolated. Estimate the number of them.

(7) Prove that $\mathbb{A}^2 \setminus (0,0)$ is not isomorphic to an affine variety. .