MATH IN MOSCOW. ALGEBRAIC GEOMETRY. HOMEWORK 6

(1) Let V be a vector space, and $W, W' \subset V$ be a vector subspaces such that $V = W \oplus W'$. Prove that the map

$$\pi \colon \mathbb{P}(V) \setminus \mathbb{P}(W) \to \mathbb{P}(W')$$

induced by the projection $V \to W'$ along W is regular. This map is called a *projection* from $\mathbb{P}(W)$ onto $\mathbb{P}(W')$. If $\dim W = 1$, we call π a *projection from a point.*

(2) Let C be a rank 3 conic in $\mathbb{P}^2$. Consider the projection $\pi \colon \mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^1$ from a point $p \notin C$, and consider its restriction $\pi|_C \colon C \to \mathbb{P}^1$. How many points are there in the preimage $\pi|_C^{-1}(x)$ of any point in $x \in \mathbb{P}^1$?

(3) Let C be a rank 3 conic in $\mathbb{P}^2$. Consider the projection $\pi \colon \mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^1$ from a point $p \in C$, and consider its restriction $\pi|_{C\setminus p} \colon C \setminus p \to \mathbb{P}^1$. Prove that $\pi|_{C\setminus p}$ can be extended to regular map $\overline{\pi}_C \colon C \to \mathbb{P}^1$, and show that $\overline{\pi}_C$ is an isomorphism.

(4) Prove that, over the field of rational numbers $\mathbb{Q}$, a rank 3 conic $C \subset \mathbb{P}^2_{\mathbb{Q}}$ is either empty, or isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$.

(5) Using previous problem , find a parametrisation for the set of Pythagorean triples, that is, integers (x, y, z) such that $x^2 + y^2 = z^2$.

(6) Similar to problem 3, construct a projection from a point p on a general cubic curve in $\mathbb{P}^2$.

(7) Prove that over $\mathbb{C}$, a general curve $C \subset \mathbb{P}^2$ is a compact oriented topological surface. Hence, by the classification theorem in topology, it is a sphere with g handles. Find g for a general cubic curve. What about a smooth curve of arbitrary degree $d \geq 1$?