MATH IN MOSCOW. ALGEBRAIC GEOMETRY. HOMEWORK 4

(1) Define a Veronese map $v_d\colon \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{n}-1}$ by the formula

$$v_d(x_0 : \ldots : x_n) \mapsto (x_0^d : x_0^{d-1}x_1 : \ldots : x_n^d)$$

(take all monomials of degree d in x_i). Prove that its image is a projective variety, and v_d is an isomorphism onto its image.

(2) Using Segre map $s\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ given by the formula

$$s[(x_0 : \ldots : x_n), (y_0 : \ldots : y_m)] \mapsto (x_0y_0 : x_0y_1 : \ldots : x_ny_m)$$

define the structure of a projective variety on $\mathbb{P}^n \times \mathbb{P}^m$.

(3) Prove that the image of the Veronese map is not contained in a hyperplane (recall that a hyperplane is the projective subspace $\mathbb{P}^{n-1}$ linearly embedded into the projective space $\mathbb{P}^n$). Prove that the image of the Segre map is not contained in a hyperplane.

(4) Prove that a subset $V \subset \mathbb{P}^n \times \mathbb{P}^m$ is closed if and only if V is given by a family of equations

$$F_i(x_0, \ldots, x_n, y_0, \ldots, y_m) = 0$$

where F_i is a polynomial homogeneous both in x_i and y_j. Formulate and prove a similar claim for closed subsets of $V \subset \mathbb{P}^n \times \mathbb{A}^m$.

(5) Let V be a 3-dimensional vector space. Denote by $\mathbb{P}(S^2V^*) = \mathbb{P}^5$ the projectivization of the space of homogeneous polynomials of dergee 2 on V. Prove that the closed subset of $\mathbb{P}(S^2V^*)$ that corresponds to plane conics of rank 1 is isomorphic to $v_2(\mathbb{P}^2)$ where v_2 is the Veronese map defined in problem 1.