MATH IN MOSCOW. ALGEBRAIC GEOMETRY. HOMEWORK 8

(1) Let $T_pX = (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$ be the Zariski tangent space to a point $p \in X$ where X is an affine algebraic set. Prove that if $X = X_1 \cup X_2$, and $p \in X_1 \cap X_2$, then $T_pX_1 \cup T_pX_2 \subset T_pX$. Is it true that $T_pX_1 \cup T_pX_2 = T_pX$?

(2) Prove that if X is non-singular and Y is non-singular then $X \times Y$ is non-singular as well.

(3) Prove that for a quasi-projective variety X and a point $p \in X$, for the local ring $\mathcal{O}_{p,X}$ we have $\mathcal{O}_{p,X} = \bigcup_{p \in U} \mathbb{K}[U]$ where U are affine neighbourhoods of p.

(4) * Prove that any rational map from a smooth curve to a projective variety can be extended to a regular map. Is it true for non-projective varieites?

(5) An algebra is called *local*, if it has a unique maximal ideal. Prove that the algebra $A = \mathbb{K}[x]/(x^n)$ is local for any $n \geq 1$.

(6) Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal such that $(x_1, \ldots, x_n)^m \subset I$ for some $m \geq 1$. Prove that the algebra $\mathbb{K}[x_1, \ldots, x_n]/I$ is finite-dimensional and local, that is, it contains a unique maximal ideal.