MATH IN MOSCOW. TOPOLOGY-2. HOMEWORK 1.
DUE TO 14 SEPTEMBER 2022

(1) Can a plane $\mathbb{R}^2$ be retracted to a unit circle $S^1 \subset \mathbb{R}^2$ with center at the origin? Same question for $S^1 \subset \mathbb{R}^2 \setminus \{0\}$.

(2) (Hopf fibration) Prove that the following map is continuous, and each fiber is homeomorphic to a circle:
$$\mathbb{C}^2 \ni (z_1, z_2) \mapsto (z_1 \colon z_2) \in \mathbb{C}P^1.$$

(3) Prove that the two definitions of the complex projective space $\mathbb{C}P^n$ are equivalent: $\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times = S^{2n+1}/S^1$.

(4) Prove that $\mathbb{R}P^1 \simeq S^1$. Prove that $\mathbb{C}P^1 \simeq S^2$. Is it true that $\mathbb{R}P^2 \simeq S^2$?

(5) Define the groups $\mathrm{PGL}(n, \mathbb{C})$ and $\mathrm{PSL}(n, \mathbb{C})$ by the following exact sequences:
$$1 \to \mathbb{C}^\times \to \mathrm{GL}(n, \mathbb{C}) \to \mathrm{PGL}(n, \mathbb{C}) \to 1,$$
$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathrm{SL}(n, \mathbb{C}) \to \mathrm{PSL}(n, \mathbb{C}) \to 1.$$
Is it true that $\mathrm{PGL}(n, \mathbb{C}) \simeq \mathrm{PSL}(n, \mathbb{C})$? Is it still true if we replace $\mathbb{C}$ by $\mathbb{R}$?

(6) Prove that the action of $\mathrm{PGL}(n+1, \mathbb{C})$ on $\mathbb{P}^n$ is transitive. Moreover, show that it maps any $n+2$ general points to any other $n+2$ general points. Explain the meaning of the word "general" in this context.